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Financing imports, the Triffin dilemma and more

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Abstract

This monograph presents a formal proof of the notion by which a country devoid of tradable assets and without access to foreign borrowing and lending must systematically pay for its imports in foreign currency through its exports alone, provided a demand for them to begin with. It likewise sets forth a formal proof of the Triffin dilemma, by which a country whose external currency enjoys the status of an international reserve currency is bound to incur a trade deficit and an attendant excess of extant foreign net borrowing in relation to its tradable assets, meanwhile advancing an innovative, orderly model of the balance of payments. Currency regimes, sudden stops in foreign net borrowing, international reserve currencies and changes in private and public consumption are additionally examined. This monograph completes its study of the dynamics pertaining to exports and foreign borrowing by means of a static deterministic partial equilibrium (SDPE) model, via stability analysis.

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MSC codes: 91B60; 91B52; 91B64.
Keywords: balance of payments; exports; imports; international reserve currency; Triffin dilemma; tradable assets.

1. Introduction

1.1 Gas imports in Roubles. Following the eruption of the Russo-Ukrainian conflict of February 2022 the North Atlantic Treaty Organization (NATO) bloc began enacting a series of sanctions against the country of Russia, whereby the European Union (EU)'s member states in particular might gradually interrupt their imports of Russian gas, in spite of its irreplaceability at current EU technology.

Prior to such the NATO bloc had additionally deployed a series of seizures of Russian assets abroad, in parts of the NATO bloc territory. Russia's response was a demand that all gas still purchased from it might be paid in its domestic currency, Roubles, and no longer in American Dollars, Euros or other reserve currencies.

1.2 Paying for imports. The broader question arising therefrom is therefore the following: how are the private and public sectors of a country (i.e. households, firms, including financial intermediaries, and government) to pay for their imports and how do they do so in practice? In simpler terms, how does a country pay for its imports, in the abstract and concrete? With the currency elected by the country from which it imports (e.g. importer currency, reserve currency, exporter currency).

Suppose the country from which it imports have declined the exporter's currency. How is such a currency, being that elected by the country from which it imports, initially acquired? By buying it on the foreign exchange market, at some nominal exchange rate (e.g. spot, future).

With what is it purchased in turn? Net of other domestic and foreign assets, which are all ultimately convertible into respective currencies for taxation or consumption purposes, as are all traded currencies, it is purchased with either domestic currency or some other foreign currency functioning as a store of value (i.e. another reserve currency), other than an exchange medium and an account unit.

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1.3 Foreign currency shortage. What if a country is wanting in sufficient reserve currencies, all other tradable assets and hardly attractive to foreign lenders, as would be an emerging country seeking to increase its imported commodities from a developed one? Would such a country manage to purchase the desired foreign currency with domestic currency alone? It would insofar as domestic currency were demanded to the end of purchasing its own exports abroad[1].

To better see this, suppose such a country lacked some of the desired foreign currency to pay for its imports, having acquired the rest by means of its exports (which can surely be null as well). At constant domestic and foreign prices, in order to purchase its imports such a country could either (i) provoke a domestic nominal exchange rate appreciation which may allow its foreign currency to potentially finance all of its imports, by purchasing domestic currency with the desired foreign currency on the foreign exchange market or (ii) purchase the required portion of the desired foreign currency with domestic currency on the foreign exchange market.

The former option would be contradictory inasmuch as the country might precisely lack the required portion of the desired foreign currency: if it possessed it in the first place it would directly finance its imports therewith. The latter option would result in an expansion of domestic currency whereby the domestic nominal exchange rate would further depreciate and although the full purchase of imports may be temporarily attained through the depreciation in question the foreign exchange market would eventually anticipate a perduring domestic nominal exchange rate depreciation and thereby decrease its demand for domestic currency, sufficiently causing the domestic nominal exchange rate to appreciate in order for imports to be fully purchased at the value of exports in foreign currency.

1.4 Example. Domestic real exchange rate: $\frac{p^{-1}x}{p^{*-1}x^{*}} = \frac{(1^{-1})2_{USD}}{(1^{-1})1_{GBP}} = \frac{2_{USD}}{1_{GBP}}$. Imports: 10 units. Value of imports in domestic currency: $\frac{2_{USD}}{1_{GBP}}(10) = 20_{USD}$. Value of imports in foreign currency: $\frac{2_{USD}}{1_{GBP}} = \frac{20_{USD}}{y_{GBP}} \longrightarrow y = 10_{GBP}$. Suppose a country only had 5_{GBP}, being the value of exports in foreign currency, namely, $\frac{1_{GBP}}{2_{USD}}(z) = 5_{GBP} \longrightarrow z = 10$ units of exports. There is a shortage of 10 units of exports for the full importation to happen, whereby $\frac{1_{GBP}}{2_{USD}}(z) = 10_{GBP} \longrightarrow z = 20$ units of exports in total.

Option 1. Domestic real exchange rate appreciation: $\frac{1_{USD}}{1_{GBP}}$, say by buying 10_{USD} with 5_{GBP} on the foreign exchange market. Such is a subtraction of 10 USD from the foreign exchange market through the addition of 5 GBP thereto, surmised to cause a 50% decrease in the USD-GBP exchange rate. It is impossible, for 5_{GBP} are already needed for half of the importation and 5_{GBP} more would be needed at the new real exchange rate: $\frac{1_{USD}}{1_{GBP}}(10) = 10_{USD}$ *qua* imports value in domestic currency and $\frac{1_{USD}}{1_{GBP}} = \frac{10_{USD}}{y_{GBP}} \longrightarrow y = 10_{GBP}$ *qua* imports value in foreign currency, thereby fully lacked, since $5_{GBP} + (-5_{GBP}) = -10_{USD} + 10_{USD} = 0$.

Option 2. Domestic real exchange rate depreciation: buy 5_{GBP} with 10_{USD}, causing $\frac{3_{USD}}{1_{GBP}}$. Such is an addition of 10 USD to the foreign exchange market through the subtraction of 5 GBP therefrom, causing a corresponding 50% increase in the USD-GBP exchange rate. It temporarily permits the full importation, that is, the value of imports in foreign currency remains unvaried: $\frac{3_{USD}}{1_{GBP}}(10) = 30_{USD}$ *qua* imports value in domestic currency and $\frac{3_{USD}}{1_{GBP}} = \frac{30_{USD}}{y_{GBP}} \longrightarrow y = 10_{GBP}$ *qua* imports value in foreign currency, thereby fully possessed, since $5_{GBP} + 5_{GBP} = 10_{GBP} \longleftrightarrow -10_{USD} + (-10_{USD}) = -20_{USD}$[2].

If the domestic real exchange rate depreciation were systematic then the foreign exchange market would adjust to it by decreasing its demand for domestic currency and thereby cause a real exchange rate appreciation whereby the value of exports in foreign currency would finance that of imports: $\frac{1_{USD}}{1_{GBP}}(10) = 10_{USD}$ *qua* imports and exports value in domestic currency and $\frac{1_{USD}}{1_{GBP}} = \frac{10_{USD}}{y_{GBP}} \longrightarrow y = 10_{GBP}$ *qua* imports and exports value in foreign currency, whence $\frac{1_{GBP}}{1_{USD}}(z) = 10_{GBP} \longrightarrow z = 10$ units of exports.

1.5 Export led importation. Otherwise phrased, how would a country devoid of tradable assets and without access to foreign borrowing and lending systematically pay for its imports in foreign currency? The banal answer is that it would systematically pay for its imports through its exports alone, provided a demand for them to begin with.

[1] Such clarifies the following clause of Saccal [14]: "... under domestic financial closure and a domestic double currency there are two options: domestic external nominal money supply M_{SE} is retained for domestic imports; ... ".

[2] Theoretically possible too: $(5_{GBP} + 5_{GBP}) + 10_{GBP} = 20_{GBP} \longleftrightarrow [-10_{USD} + (-10_{USD})] + (-30_{USD}) = -50_{USD}$; $\frac{1_{GBP}}{3_{USD}}(z) = 10_{GBP} \longrightarrow z = 30$ units of exports.

More generally, therefore, a country devoid of real tradable assets and without access to foreign real borrowing and lending possesses sufficient foreign currency to systematically pay for its imports if and only if the real money value of its exports is no less than that of its imports. This monograph formalises such a condition, in a static or continuous time environment. It additionally formalises the Triffin dilemma and innovatively presents an orderly model of the balance of payments, however stylised.

It finally studies optimal exportation and foreign borrowing by means of an SDPE model, making further sense of the said formalisations. Related works are Hemphill [9], Winters [18], Khan [10], Mazarei [12], Moran [13], Khan and Knight [11], Antzoulatos and Peart [1], Edwards [7], Bordo and McCauley [4], Bordo and McCauley [5], Bordo and McCauley [6] and Farhi and Maggiori [8]. As the case thus far, comparative statics or dynamics all through the rest of this monograph relinquish the explicit usage of the "all else equal" clause.

2. Framework

2.1 Countries and currencies. Envisage a world with two countries engaging in trade. Refer to country 1 as the domestic country and country 2 as the foreign country. Let the domestic country feature an internal currency M_{SI}, used for internal balances (i.e. monetary policy[3], full employment, price stability), and an external currency M_{SE}, used for external balances (i.e. exchange rate policy, trade).

Such signifies that domestic internal currency M_{SI} cannot be traded on the foreign exchange market, but on the domestic exchange one (i.e. domestic securities market) alone, and only circulates within the domestic country's confines, under internal financial closure (i.e. capital controls, capital account inconvertibility); domestic external currency M_{SE} cannot be specularly traded on the domestic exchange market, but on the foreign exchange market alone, and only circulates without the domestic country's confines.

For completeness, internal balances are also affected by prices and external balances are also affected by trade policy, encompassing the entire balance of payments (i.e. current account and capital account and sometimes, on separate itemisation, even financial account).

Let the foreign country only feature one currency M_S^*, used for both internal and external balances and thereby admitting foreign sterilisation. Exclude currency substitutions (e.g. dollarisation) on either side: $M_{SI},\ M_{SE},\ M_S^* \in \mathbb{R}_{++}$.

Domestic internal and external currencies M_{SI} and M_{SE} and foreign currency M_S^* are therefore respectively understood as being (i) domestic internal nominal money supply, (ii) domestic external nominal money supply and (iii) foreign nominal money supply.

2.2 Sterilisation. In general, under a single currency and internal financial openness at home and abroad, whatever the nominal exchange rate regime between the two countries' currencies, a domestic sterilisation is a currency intervention by the domestic central bank on the foreign exchange market such that an increase (or decrease) in domestic currency through the purchase (or sale) of foreign currency or foreign securities, to the end of a domestic nominal exchange rate depreciation (or appreciation), is neutralised by a decrease (or increase) in domestic currency through the sale (or purchase) of domestic securities, conducted by the domestic central bank on the domestic securities market (i.e. one with the foreign securities market and that of foreign exchange).

The permanence of the change in the nominal exchange rate between the two countries' currencies is debated. Strictly speaking, the volume of domestic currency remains unchanged and so does the nominal

[3] Saccal [16] wrote: "... anti-cyclical monetary policy (i.e. neutralisation of internal temporary supply shocks and, sub-optimally, also of demand).". Strictly speaking, however, internal and external temporary supply shocks are to be primarily neutralised through supply sided fiscal policy (i.e. firm subsidies and firm cyclical taxation), concerning production, just as internal and external temporary demand shocks are to be primarily neutralised through demand sided fiscal policy (i.e. cyclical government spending, household subsidies and household cyclical taxation), concerning consumption. Internal temporary supply and demand shocks are to be therefore secondarily neutralised through monetary policy and external temporary supply and demand shocks secondarily through exchange rate policy, with a debatably greater implementation immediacy both internally and externally. Internal and external permanent supply shocks cannot be neutralised and internal permanent demand shocks do not exist by definition. External permanent demand shocks involve growth mismatches in multinational external real output and external real money supply, under trade (i.e. Rodrik's trilemma, as per Saccal [16]), and can only be resolved through autarky or multinational external real money supply growth equality (i.e. neutralisation), *ex ante* or *ex post;* in the words of Thirlwall [17]: "... it is more difficult for a country to rectify an import-export gap than it is to rectify a savings-investment gap.".

exchange rate between the two countries' currencies, but the domestic and foreign private sectors could respond to the domestic sterilisation by adjusting their portfolio balances in turn, in accordance with their degree of substitution between domestic securities and foreign securities, including currency, or by updating their expectations on the nominal exchange rate between the two countries' currencies, thereby affecting it, through the channel of demand.

2.3 Double currency. The nature of the nominal exchange rate between domestic external currency M_{SE} and foreign currency M_S^* is either fixed, managed (e.g. crawling) or floating and it determines whether domestic external currency M_{SE} be traded on the foreign exchange market or not.

In the cases of a managed and floating nominal exchange rate between domestic external currency M_{SE} and foreign currency M_S^* the same is only established by means of Open Market Operations (OMOs) on the foreign exchange market, whereby domestic external currency M_{SE} is traded indeed.

In the case of a fixed nominal exchange rate between domestic external currency M_{SE} and foreign currency M_S^* the same can be either established by means of OMOs or decreed. In the event it were decreed all exchange of foreign currency M_S^* for domestic external currency M_{SE} would pass through the domestic central bank, under external financial closure. The foreign importer would then exchange foreign currency M_S^* for domestic external currency M_{SE} through the domestic central bank for domestic exportation.

Under a double currency and OMOs on the foreign exchange market (i.e. non-decree fixed, managed or floating nominal exchange rate between the two currencies and external financial openness) domestic external currency M_{SE} would be held by the domestic central bank and the foreign exchange market, while under a double currency and a decree fixation of the nominal exchange rate between domestic external currency M_{SE} and foreign currency M_S^* (i.e. decree fixed nominal exchange rate between the two currencies and external financial closure) domestic external currency M_{SE} would be held by the domestic central bank alone.

Table 1: Currency regimes

	Domestic nominal money supply (currency units)		Foreign nominal money supply (currency units)	Means	Domestic nominal money supply (currency units)		Foreign nominal money supply (currency units)	Means
External balances (real exchange rate)	$x_{M_{SE}}$	:	$x^*_{M_S^*}$	Decree, OMOs	x_{M_S}	:	$x^*_{M_S^*}$	Decree
Internal balances (real money supply)	$x_{M_{SI}}$			OMOs	x_{M_S}			OMOs

Note. This table presents the two currency regimes a country is to choose from in the regards of its external balances whenever adopting fixed, managed or floating nominal exchange rates under internal financial closure: (i) domestic external currency M_{SE}, exchangeable for foreign currency M_S^*, and domestic internal currency M_{SI} (i.e. double currency, e.g. Cuban convertible Peso and non-convertible Peso); (ii) domestic currency M_S, exchangeable for foreign currency M_S^* (i.e. single currency, e.g. Chinese Yuan). In the first case domestic external currency M_{SE} is exchangeable for foreign currency M_S^* at nominal exchange rate $E = x^{*\,-1}_{M_S^*} x_{M_{SE}}$ established by decree or OMOs on the foreign exchange market, given domestic external prices p_E and foreign prices p^*, while domestic internal currency M_{SI} is used for internal balances and regulated through OMOs on the domestic securities market, given positive domestic internal prices p_I. In the second case domestic currency M_S is exchangeable for foreign currency M_S^* at nominal exchange rate $E = x^{*\,-1}_{M_S^*} x_{M_S}$ established by decree alone, given domestic and foreign prices p and p^*, for domestic currency M_S is also used for internal balances and regulated through OMOs on the domestic securities market. Under a double currency domestic external currency M_{SE} is converted into domestic internal currency M_{SI} at a fixed, managed or floating rate, by means of a decree or OMOs, as required. Under a single currency a portion of domestic currency M_S is purposely stored by the domestic central bank for domestic exportation.

2.4 Single currency. If the domestic country only featured one currency M_S as well and were under internal financial closure then it could not be traded on the foreign exchange market, but would be exchanged for foreign currency M_S^*, at a fixed nominal exchange rate between them, by passing through the domestic central bank whenever needed for domestic exportation.

The foreign importer (i.e. domestic exportation, foreign importation) would exchange foreign currency M_S^* for domestic currency M_S through the domestic central bank, whereby a portion of domestic currency M_S would be purposely stored by the domestic central bank for domestic exportation.

If the foreign country were under internal financial closure and were the one to fix the nominal exchange rate between domestic currency M_S and foreign currency M_S^* by decree, instead of the domestic country, then the domestic importer (i.e. domestic importation, foreign exportation) would accordingly exchange domestic currency M_S for foreign currency M_S^* through the foreign central bank.

If the nominal exchange rate between domestic currency M_S and foreign currency M_S^* were not fixed by decree, but were either fixed through OMOs, managed or floating, under internal financial openness, then both the domestic exporter (after sale) or foreign importer (before purchase) and the foreign exporter (after sale) or domestic importer (before purchase) would exchange foreign currency M_S^* and domestic currency M_S for one another (i) primarily through OMOs on the foreign exchange market and (ii) secondarily by passing through the reciprocal central banks.

In other words, under a single currency, internal financial openness and a floating, managed or fixed nominal exchange rate between them, granted sufficient foreign reserves on both ends, the domestic exporter (after sale) or foreign importer (before purchase) could exchange foreign currency M_S^* for domestic currency M_S through OMOs on the foreign exchange market or through both the domestic and foreign central banks, just as the foreign exporter (after sale) or domestic importer (before purchase) could exchange domestic currency M_S for foreign currency M_S^* through the same channels.

A single currency, internal financial openness, a fixed or managed nominal exchange rate and the exchange of the two currencies through OMOs on the foreign exchange market would obviously call for a currency intervention by the central bank committed to the nominal exchange rate fixation.

Table 2: Impossibility of a double currency for the domestic real trade balance

	Domestic nominal money supply (currency units)		Foreign nominal money supply (currency units)	Means
Domestic real exports (real exchange rate)	$x_{M_{SI}}$	:	$x^*_{M_S^*}$	Decree
Domestic real imports (real exchange rate)	$x_{M_{SE}}$	:	$x^*_{M_S^*}$	OMOs

Note. This table presents the impossible scenario of a country adopting a double currency for the domestic real trade balance (i.e. net real exports): (i) domestic internal currency M_{SI} exchangeable for foreign currency M_S^* in the regards of domestic real exports ex; (ii) domestic external currency M_{SE} exchangeable for foreign currency M_S^* in the regards of domestic real imports im. In the regards of domestic real exports ex domestic internal currency M_{SI} would be exchangeable for foreign currency M_S^* at nominal exchange rate $E = x^{*-1}_{M_S^*} x_{M_{SI}}$ established by decree alone, given domestic internal prices p_I and foreign prices p^*, for domestic internal currency M_{SI} would also be used for internal balances and regulated through OMOs on the domestic securities market; indeed, a portion of domestic internal currency M_{SI} would be purposely stored by the domestic central bank for domestic exportation. In the regards of domestic real imports im domestic external currency M_{SE} would be exchangeable for foreign currency M_S^* at nominal exchange rate $E = x^{*-1}_{M_S^*} x_{M_{SE}}$ established by OMOs on the foreign exchange market, given domestic external prices p_E and foreign prices p^*. The foreign exchange market would at some point discern domestic external currency M_{SE} to be worthless and inutile for all consumption and taxation revenue convertibility and would thereby reject it. In fact, if in the regards of domestic real imports im domestic external currency M_{SE} were exchangeable for foreign currency M_S^* at nominal exchange rate $E = x^{*-1}_{M_S^*} x_{M_{SE}}$ established by decree, as opposed to OMOs, the rejection process at the expense of domestic external currency M_{SE} by the foreign exchange market would then unfold with considerably greater speed.

2.5 Currency regimes. Regardless of whether the domestic country featured external currency M_{SE} and internal currency M_{SI} or only one currency M_S, if both the domestic and the foreign country sought a fixed nominal exchange rate between their respective currencies, that is, between domestic external currency M_{SE} or domestic currency M_S and foreign currency M_S^*, and: (i) if both sought it via decree or OMOs then current and capital account trade and exchange between them would ultimately not eventuate; (ii) if only one sought via decree then the above scenario would be in act.

By contrast, if (i) for domestic exportation the domestic country were to employ domestic internal currency M_{SI} under internal financial closure at a domestic nominal exchange rate between domestic internal currency M_{SI} and foreign currency M_S^* fixed by decree and (ii) for domestic importation the domestic country were to employ domestic external currency M_{SE} under internal financial closure at a domestic nominal exchange rate between domestic external currency M_{SE} and foreign currency M_S^* managed through OMOs on the foreign exchange market, targeting appreciative policies, the foreign exchange market would then eventually internalise the inutility of domestic external currency M_{SE} for taxation or consumption purposes and reject it.

Such a rejection would be analogous to that of cryptocurrencies, which, unlike the hypothetical domestic external currency M_{SE} treated hereby, are not even legal tender (more anon). The domestic country would thus have to regress to one of the two aforesaid currency regimes and in all events finance domestic importation through (i) tradable assets (e.g. foreign reserves, precious metals, demand and supply inelastic securities), (ii) foreign borrowing and lending or (iii) domestic exportation.

2.6 External and internal currencies. Domestic external currency M_{SE} is converted into domestic internal currency M_{SI} either (i) at a fixed rate through the domestic central bank, by means of a decree, or (ii) through OMOs on the domestic exchange market.

In the former case the nominal exchange rate between domestic external currency M_{SE} and domestic internal currency M_{SI} would indeed be fixed and all domestic external currency M_{SE} would only be held by the domestic exporter or the domestic central bank.

In the latter case the nominal exchange rate between domestic external currency M_{SE} and domestic internal currency M_{SI} would be managed or floating and domestic external currency M_{SE} would be held by the domestic private sector, domestic exporter included, or the domestic central bank.

A higher nominal exchange rate between domestic external currency M_{SE} and domestic internal currency M_{SI} would incentivise domestic importation, affording a relatively higher amount of foreign currency M_S^* in terms of domestic internal currency M_{SI}.

A lower nominal exchange rate between the two does not incentivise domestic exportation, however, but merely discourages domestic importation, affording a relatively lower amount of foreign currency M_S^* in terms of domestic internal currency M_{SI}. Domestic exportation would be incentivised through its identification with foreign importation in a condition whereby the two countries inverted roles.

2.7 Double internal currency. Two internal currencies not functioning as precisely one, either being thereby redundant, would by contrast eventually fail, for one would end up being demanded less than the other to the point of complete rejection.

The way by which two internal currencies would function as one internal currency involves taxation and consumption convertibility equivalence, that is, at given prices, both internal currencies should be convertible into taxation revenue and consumption at the same nominal value. If either currency exhibited taxation revenue non-convertibility then such a one would not be legal tender nor money. If either currency exhibited consumption non-convertibility then such a one would be rejected outright, failing convertibility into taxation revenue and correspondingly being neither legal tender nor money.

Legal tender has historically fallen under the following monetary classifications: non-representative or fiat money; representative money, of an underlying commodity or asset, sufficiently liquid and demand and supply inelastic; commodity or asset money. In order to become legal tender money canonically classified (i.e. store of value, unit of account, medium of exchange) would have to be decreed as a legal form of (debt re)payment across all national (debt) transactions.

2.8 Exchange rates. Let a unit of domestic external currency x and one of foreign currency x^* determine the domestic nominal exchange rate E and the foreign nominal exchange rate $E^* : \ x \in M_{SE}$ and $x^* \in M_S^*$ such that $E = \frac{x}{x^*}$ and $E^* = \frac{1}{E} = \frac{x^*}{x}$. Accordingly, for positive domestic external prices p_E and foreign prices p^*, there arise domestic and foreign real exchange rates e and $e^* : \ \forall p_E, \ p^* \in \mathbb{R}_{++}, \ e = \frac{p_E^{-1} x}{p^{*-1} x^*} = \frac{x p^*}{x^* p_E}$ and $e^* = \frac{1}{e} = \frac{x^* p_E}{x p^*}$.

As outlined, the domestic country's nominal exchange rate E can be either fixed, managed or floating: in the latter two cases it is only established by means of OMOs on the foreign exchange market, while in the former case it can be also decreed.

2.9 Real money supplies and money demands. Domestic external real money supply m_{SE} is the quotient of domestic external currency M_{SE} and domestic external prices $p_E : \ m_{SE} = \frac{M_{SE}}{p_E}$. Foreign real money supply m_S^* is the quotient of foreign currency M_S^* and foreign prices $p^* : \ m_S^* = \frac{M_S^*}{p^*}$.

Absent loss of generality, domestic external money demand m_{DE} is an increasing function of positive domestic export demand ed and a decreasing function of positive domestic import demand $id : \ \forall ed, \ id \in$

$\mathbb{R}_{++},\ m_{DE} = f(\overset{+}{ed},\ \overset{-}{id})$.

Foreign money demand m^*_D, theoretically also affected by foreign demand and supply taxation, foreign marginal products, foreign technology and foreign output demand, inclusive of foreign export and import demand, is instead for simplicity an increasing function of positive foreign autonomous money demand $am^*_D:\ \forall am^*_D \in \mathbb{R}_{++},\ \ m^*_D = f(\overset{+}{am^*_D})$.

2.10 Real interest rates and expectations. Domestic external real interest rate r_E is a decreasing function of domestic external real money supply m_{SE} and an increasing function of domestic external money demand $m_{DE}:\ r_E = f(\overset{-}{m_{SE}},\ \overset{+}{m_{DE}})$. Foreign real interest rate r^* is a decreasing function of foreign real money supply m^*_S and an increasing function of foreign money demand $m^*_D:\ r^* = f(\overset{-}{m^*_S},\ \overset{+}{m^*_D})$.

Domestic external expected real interest rate er_E is a decreasing function of domestic external real interest rate r_E and an increasing function of a domestic dummy variable ε suitably accommodating long run stabilisations: $er_E = f(\overset{-}{r_E},\ \overset{+}{\varepsilon})$.

In detail, domestic dummy variable ε is a decreasing function of long run domestic external real money supply $m_{SE_{LR}}$ and of long run domestic external money demand $m_{DE_{LR}}:\ \forall m_{SE_{LR}},\ m_{DE_{LR}} \in \mathbb{R}_+,\ \varepsilon = f(\overset{-}{m_{SE_{LR}}},\ \overset{-}{m_{DE_{LR}}})$.

Such is because the foreign exchange market adjusts its demand for domestic external real money supply m_{SE} or for domestic external money demand m_{DE} whenever either one undergo a permanent variation, as outlined above. Naturally, changes in long run domestic external real money supply $m_{SE_{LR}}$ and in long run domestic external money demand $m_{DE_{LR}}$ are permanent and thereby suggest ones in their short run correlatives too, domestic external real money supply m_{SE} and domestic external money demand m_{DE}, that is to say, but not the converse, being temporary.

Foreign expected real interest rate er^* is a decreasing function of foreign real interest rate r^* and an increasing function of a foreign dummy variable ε^* suitably accommodating long run stabilisations: $er^* = f(\overset{-}{r^*},\ \overset{+}{\varepsilon^*})$. Analogously, foreign dummy variable ε^* is a decreasing function of long run foreign real money supply $m^*_{S_{LR}}$ and of long run foreign money demand $m^*_{D_{LR}}:\ \forall m^*_{S_{LR}},\ m^*_{D_{LR}} \in \mathbb{R}_+,\ \varepsilon^* = f(\overset{-}{m^*_{S_{LR}}},\ \overset{-}{m^*_{D_{LR}}})$.

2.11 Real exchange rates. Domestic real exchange rate e is thence a decreasing function of domestic external real interest rate r_E and foreign expected real interest rate er^* and an increasing function of foreign real interest rate r^* and domestic external expected real interest rate $er_E:\ e = f(\overset{-}{r_E},\ \overset{+}{er_E},\ \overset{-}{er^*},\ \overset{+}{r^*})$.

Foreign real exchange rate e^* is likewise a decreasing function of foreign real interest rate r^* and domestic external expected real interest rate er_E and an increasing function of domestic external real interest rate r_E and foreign expected real interest rate $er^*:\ e^* = f(\overset{-}{r^*},\ \overset{+}{er^*},\ \overset{-}{er_E},\ \overset{+}{r_E})$.

3. Importation finance conditions

Proposition 3.1 (Importation finances) *Let the domestic country possess no real tradable assets nor access to foreign real borrowing and lending. Let domestic net real output receipts and domestic net real transfer receipts be null. Let domestic real exports be non-negative and domestic real imports be positive. The domestic real balance of payments, free of statistical discrepancies, is then characterised by the domestic real trade balance. Formally:*

$$\begin{aligned}
&(\pi_E\dot{b} - \dot{e}\cdot\dot{b}^*) - ta + ex + (ny_r + n_2tr_r) = e\cdot im + (rb - e\cdot r^*b^*) \longrightarrow \\
&\longrightarrow (\pi_E\dot{b} + e\cdot r^*b^*) - (\dot{e}\cdot\dot{b}^* + rb) - ta + ex + (ny_r + n_2tr_r) = e\cdot im \longrightarrow \\
&\longrightarrow (ex - e\cdot im) + (ny_r + ntr_r) = (\dot{e}\cdot\dot{b}^* - \pi_E\dot{b}) + ta \longrightarrow nx + ny_r + ntr_r = nl + ta \longrightarrow \\
&\longrightarrow bop:\ ca = ca_2 \longrightarrow \forall ta,\ \dot{b},\ b,\ \dot{b}^*,\ b^*,\ ny_r,\ ntr_r = 0,\ ex\in\mathbb{R}_+,\ im\in\mathbb{R}_{++},\ bop:\ ex = e\cdot im,
\end{aligned}$$

where $\pi_E = \frac{\dot{p}_E}{p_E}$, $\dot{e} = \frac{x\dot{p}^*}{\dot{x}^*p_E}$ *and* $n_2tr_r = ntr_r - (e\cdot r^*b^* - rb)$. *Indeed, (i)* $\pi_E\dot{b}$, *since* $\frac{\dot{p}_E\dot{b}}{p_E}$, *(ii)* $\dot{e}\cdot\dot{b}^*$, *since* $\frac{\dot{E}\dot{p}^*\dot{b}^*}{p_E} = \left(\frac{x\dot{p}^*}{\dot{x}^*}\right)\frac{\dot{b}^*}{p_E}$, *(iii)* $e\cdot r^*b^*$, *since* $\frac{Ep^*r^*b^*}{p_E} = \left(\frac{xp^*}{x^*}\right)\frac{r^*b^*}{p_E}$, *(iv)* $e\cdot im$, *since* $\frac{Ep^*im}{p_E} = \left(\frac{xp^*}{x^*}\right)\frac{im}{p_E}$, *and (v)*

$\forall x = ta,\ ex,\ ny_r,\ n_2tr_r,\ ntr_r,\ rb$, one has x, since $\frac{p_E x}{p_E}$. There thus exists a quantity of foreign currency for domestic importation if and only if the real money value of domestic exports is no smaller than that of domestic imports. Formally:

$$\exists n_{im}(M_S^*) \longleftrightarrow ex \geq e \cdot im.$$

Proof. Lemma 3.1.1.1 Sufficiency ($\longrightarrow$); proof by contraposition and by contradiction. Suppose $ex < e \cdot im$. Then $ex < e \cdot im = \left(\frac{Ep^*}{p_E}\right) im = \left(\frac{xp^*}{x^*p_E}\right) im = n_{im}(M_{SE})$ and since $\frac{\tilde{n}(M_{SE})}{\tilde{n}(M_S^*)} = \left(\frac{xp^*}{x^*p_E}\right) \longrightarrow \tilde{n}(M_S^*) = \left(\frac{x^*p_E}{xp^*}\right) \tilde{n}(M_{SE}) = e^*\tilde{n}(M_{SE}) = e^{-1}\tilde{n}(M_{SE})$ it follows that $ex < e \cdot im = n_{im}(M_{SE}) = e \cdot n_{im}(M_S^*) \vdash e^{-1}ex = e^*ex = \left(\frac{x^*p_E}{xp^*}\right) ex < im = e^{-1}n_{im}(M_{SE}) = e^*n_{im}(M_{SE}) = n_{im}(M_S^*) \vdash e^{-1}ex - im = e^*ex - im = e^*[n_{ex}(M_{SE}) - n_{im}(M_{SE})] = n_{ex}(M_S^*) - n_{im}(M_S^*) < 0 \vdash \nexists n_{im}(M_S^*) = im$, yielding the contraposition.

An upshot is that $im = n_{im}(M_S^*) : \ n_{im}(M_S^*) = e^{-1}n_{im}(M_{SE}) = e^*n_{im}(M_{SE}) = \left(\frac{x^*p_E}{xp^*}\right)\left(\frac{xp^*}{x^*p_E}\right) im = im \vdash e \cdot n_{im}(M_S^*) = n_{im}(M_{SE}) = \left(\frac{xp^*}{x^*p_E}\right) im$.

Similarly, $ex = n_{ex}(M_{SE}) : \ n_{ex}(M_{SE}) = e \cdot n_{ex}(M_S^*) = \left(\frac{xp^*}{x^*p_E}\right)\left(\frac{x^*p_E}{xp^*}\right) ex = ex \vdash e^{-1}n_{ex}(M_{SE}) = e^*n_{ex}(M_{SE}) = n_{ex}(M_S^*) = \left(\frac{x^*p_E}{xp^*}\right) ex$.

Lemma 3.1.1.2 Now, $ex - e \cdot im = n_{ex}(M_{SE}) - n_{im}(M_{SE}) = e[n_{ex}(M_S^*) - n_{im}(M_S^*)] < 0$ and if $\downarrow x \in \downarrow M_{SE} \vdash \downarrow n_{im}(M_{SE}) = \downarrow e \cdot n_{im}(M_S^*)$ such that $ex - e \cdot im = n_{ex}(M_{SE}) - n_{im}(M_{SE}) = e[n_{ex}(M_S^*) - n_{im}(M_S^*)] = 0$ and $\exists n_{im}(M_S^*) = im$ then because $M_{SE} \sim M_S^*$ it follows that $(\downarrow x \in \downarrow M_{SE}) = (\uparrow x^* \in \uparrow M_S^*)$, but $\uparrow x^* \in \uparrow M_S^*$ is not feasible, precisely because of $n_{ex}(M_S^*) - n_{im}(M_S^*) < 0$, and thus neither are $ex - e \cdot im = n_{ex}(M_{SE}) - n_{im}(M_{SE}) = e[n_{ex}(M_S^*) - n_{im}(M_S^*)] = 0$ and $\exists n_{im}(M_S^*) = im$, maintaining the contraposition.

Lemma 3.1.1.3 Comparably, if $\uparrow x \in \uparrow M_{SE} \vdash \uparrow n_{im}(M_{SE}) = \uparrow e \cdot n_{im}(M_S^*)$ such that $\tilde{n}_{im}(M_{SE}) \sim \tilde{n}_{im}(M_S^*) = n_{im}(M_S^*) - n_{ex}(M_S^*) > 0$, causing $\uparrow e = \frac{\uparrow xp^*}{x^*p_E}$, then $ex - e \cdot im = n_{ex}(M_{SE}) - n_{im}(M_{SE}) = e[n_{ex}(M_S^*) - n_{im}(M_S^*)] \ll 0$, but eventually $\downarrow e = \frac{\downarrow xp^*}{x^*p_E}$, since $\uparrow M_{SE_{LR}} \vdash \uparrow m_{SE_{LR}} \vdash \downarrow \varepsilon \vdash \downarrow er_E \vdash \downarrow e$, such that $ex - e \cdot im = n_{ex}(M_{SE}) - n_{im}(M_{SE}) = e[n_{ex}(M_S^*) - n_{im}(M_S^*)] \geq 0 \vdash n_{ex}(M_S^*) - n_{im}(M_S^*) \geq 0$ and $\exists n_{im}(M_S^*) = im$, yielding a contradiction in $n_{ex}(M_S^*) - n_{im}(M_S^*) \geq 0 \vdash n_{ex}(M_S^*) \geq n_{im}(M_S^*)$ relative to supposition $ex < e \cdot im \vdash e^{-1}ex = e^*ex = n_{ex}(M_S^*) < im = n_{im}(M_S^*)$.

Lemma 3.1.2 Necessity ($\longleftarrow$); direct proof. Begin with $ex \geq e \cdot im$. Then $ex \geq e \cdot im = \left(\frac{Ep^*}{p_E}\right) im = \left(\frac{xp^*}{x^*p_E}\right) im = n_{im}(M_{SE}) = e \cdot n_{im}(M_S^*)$. It follows that $ex \geq e \cdot im = n_{im}(M_{SE}) = e \cdot n_{im}(M_S^*) \vdash e^{-1}ex = e^*ex = \left(\frac{x^*p_E}{xp^*}\right) ex \geq im = e^{-1}n_{im}(M_{SE}) = e^*n_{im}(M_{SE}) = n_{im}(M_S^*) \vdash e^{-1}ex - im = e^*ex - im = e^*[n_{ex}(M_{SE}) - n_{im}(M_{SE})] = n_{ex}(M_S^*) - n_{im}(M_S^*) \geq 0 \vdash \exists n_{im}(M_S^*) = im$. *QED*

3.2 Net real exports. The difference between domestic real exports ex and domestic imports real money value $e \cdot im$ normally equals domestic net real exports nx (i.e. real trade balance): $nx = ex - e \cdot im$. As per Saccal [16], domestic tradable real output y_{TR} is itself the difference between domestic external real output y_E and domestic net real exports $nx : \ y_{TR} = y_E - nx$. Otherwise expressed, it is the sum of domestic external real output y_E and of the real money value of domestic imports $e \cdot im$, net of domestic real exports $ex : \ y_{TR} = y_E + e \cdot im - ex$.

Domestic real exports ex are an increasing function of domestic real exchange rate e and domestic export demand $ed : \ ex = f(\overset{+}{e}, \overset{+}{ed})$. Domestic real imports im are a decreasing function of domestic real exchange rate e and an increasing function of domestic import demand $id : \ im = f(\overset{-}{e}, \overset{+}{id})$.

Such two functional specifications, together with the above functional specification proper to the domestic real exchange rate e, predisposes the satisfaction of the Marshall Lerner condition for both domestic external real money supply m_{SE} and domestic external money demand m_{DE}. See Saccal [14] and Saccal [15] for more details.

Domestic external real output y_E is then an increasing function of domestic external real money supply

m_{SE} and domestic external money demand m_{DE} and a decreasing function of foreign real money supply m^*_S and foreign money demand m^*_D : $y_E = f(\overset{+}{m}_{SE},\ \overset{+}{m}_{DE},\ \overset{-}{m^*_S},\ \overset{-}{m^*_D})$.

3.3 Real current account. Domestic real current account ca is the sum of domestic net real exports nx, domestic net real output receipts ny_r and domestic net real transfer receipts ntr_r; whenever the latter two be null domestic real current account ca equals domestic net real exports nx : $ca = nx + ny_r + ntr_r \vdash \forall ny_r,\ ntr_r = 0,\ ca = nx$, as in Saccal [14] and Saccal [15].

Domestic net real output receipts ny_r and domestic net real transfer receipts ntr_r are the respective difference between (i) domestic real output receipts and payments y_r and y_p and (ii) domestic real transfer receipts (including foreign aid) and payments tr_r and tr_p : $ny_r = y_r - y_p$ and $ntr_r = tr_r - tr_p$.

3.4 Importation finances revisited. Now, the domestic supply S sum of domestic tradable real output y_{TR} and domestic non-tradable real output y_{NTR} supplies the domestic demand D of domestic real private consumption c, domestic real public consumption or government expenditure g, domestic real firm consumption or investment i, domestic real output receipts y_r and domestic real transfer receipts tr_r, net of domestic real output payments y_p and domestic real transfer payments tr_p : $S \equiv y_{TR} + y_{NTR} = c + g + i + (y_r - y_p) + (tr_r - tr_p) = c + g + i + ny_r + ntr_r \equiv D$.

If domestic non-tradable real output y_{NTR} and domestic demand D are null then domestic tradable real output y_{TR} can be rearranged such that domestic external real output y_E equals domestic net real exports nx or domestic real current account ca : $y_{NTR} = D = 0 \longrightarrow y_{TR} = y_E + e \cdot im - ex = 0 \longrightarrow y_E = nx = ex - e \cdot im = ca$.

If domestic non-tradable real output y_{NTR}, domestic real private consumption c, domestic real public consumption or government expenditure g and domestic real firm consumption or investment i alone are null then domestic tradable real output y_{TR} can be rearranged such that domestic external real output y_E equals domestic real current account ca alone: $y_{NTR} = c + g + i = 0 \longrightarrow y_{TR} = y_E + e \cdot im - ex = ny_r + ntr_r \longrightarrow y_E = ca = (ex - e \cdot im) + (y_r - y_p) + (tr_r - tr_p) = nx + ny_r + ntr_r$.

It follows that a necessary and sufficient condition for financing positive domestic real imports im in foreign currency, given non-negative domestic real exports ex, in the absence of domestic real tradable assets ta, foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$, domestic net real output receipts ny_r, domestic net real transfer receipts ntr_r, domestic real private consumption c, domestic real public consumption or government expenditure g and domestic real firm consumption or investment i, is that domestic net real exports nx, domestic real current account ca or domestic external real output y_E be non-negative: $\forall ta,\ \dot{b},\ b,\ \dot{b}^*,\ b^*,\ ny_r,\ ntr_r,\ c,\ g,\ i = 0,\ ex \in \mathbb{R}_+,\ im \in \mathbb{R}_{++},\ \exists n_{im}(M^*_S) \longleftrightarrow nx = ca = y_E \geq 0$, *ceteris paribus*. In the event the domestic country only featured one currency M_S, in addition, nothing in the above proposition and proof would substantially vary.

3.5 Savings shortage. Abstracting from domestic and foreign net real output and transfer receipts $ny_r,\ ntr_r,\ ny^*_r$ and ntr^*_r, if domestic non-tradable real output y_{NTR} and domestic demand D are not null and market clearance equation $S = D$ is rearranged into domestic national accounting identity $y = c + g + i + nx$ then the hypothetical negativity of domestic net real exports nx or domestic real current account ca, which is the excess of domestic imports real money value $e \cdot im$ relative to domestic real exports ex, is identified with a shortage of domestic real savings s in relation to domestic real firm consumption or investment i.

Domestic firms are thereby consuming more than what the country is saving, that is, than what the country is producing net of what domestic households and the domestic government (i.e. domestic treasury) are consuming, net of positive domestic taxation t; such an excess consumption by domestic firms, possibly even abroad, can be financed both at home and abroad (e.g. foreign direct investment), while the excess of domestic imports real money value $e \cdot im$ relative to domestic real exports ex is financed precisely through domestic real tradable assets ta or foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$, hereby although excluded: $\forall t \in \mathbb{R}_{++},\ y_{TR} + y_{NTR} = (y_E + e \cdot im - ex) + y_{NTR} = c + g + i \longrightarrow y_E + y_{NTR} = y = c + g + i + ex - e \cdot im = c + g + i + nx = c + g + i + ca \longrightarrow y - (c + g + i) = ex - e \cdot im = nx = ca \longrightarrow (y - c - t) + (t - g) - i = nx = ca \longrightarrow (s_h + s_g) - i = nx = ca \longrightarrow s - i = ex - e \cdot im$ such that $e \cdot im \geq ex \longleftrightarrow i \geq s$, being nonetheless necessary and sufficient for $\nexists n_{im}(M^*_S) = im$ hereby, lacking there ta and $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$.

By way of an inclusive example, in no less than the last two decades the United States of America (USA), whose real tradable assets ta are undercut by the international reserve currency status enjoyed by the American Dollar, have tended to run a trade deficit pronouncedly financed by China (more anon): on average, American $e \cdot im \geq ex \longleftrightarrow i \geq s$, despite American $c + g \gg -ca,\ i > 0$, where American $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$ from China finances American net real imports $nm \equiv -nx = -ca = e \cdot im - ex \geq 0$ especially.

3.6 Sudden stops. A further rearrangement of domestic national accounting identity $y = c+g+i+nx$ in terms of sectoral balances, as per Bagnai [2], absent domestic real tradable assets ta, similarly expounds the phenomenon of sudden stops in foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$, if initially permitted: $\forall t \in \mathbb{R}_{++},\ y = c+g+i+ca = c+g+i+nx = c+g+i+ex-e \cdot im \longrightarrow y-(c+g+i)+(e \cdot im - ex) = 0 \longrightarrow (y-c-t)-i+(t-g)+(e \cdot im - ex) = (s_h - i) + s_g + nx^* = (s_h - i) + s_g + ca^* = 0$ such that $\downarrow (s_h - i) + s_g + \uparrow nx^* = \downarrow (s_h - i) + s_g + \uparrow ca^* = 0,\ e \cdot im - ex > 0$ being financed through $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$ in positive domestic real private debt $i - s_h > 0$, until a sudden stop in $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$ whereby foreign net real exports $\downarrow nx^* = \downarrow ca^*$ such that $e \cdot im < ex$ and $\uparrow (s_h - i) + \downarrow s_g + \underbrace{nx^*}_{<0} = \uparrow (s_h - i) + \downarrow s_g + \underbrace{ca^*}_{<0} = 0$, reimbursement of extant foreign net real borrowing $rb - e \cdot r^* b^*$ being due through positive domestic real public debt $d_g = -s_g > 0$, for negative domestic real public savings $s_g < 0$, in positive domestic real private credit $s_h - i > 0$.

Victims of sudden stops in foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$ have for instance been the peripheral countries of the Euro Area (EA), such as Portugal, Greece, Spain and Ireland (see Bagnai [3]), initially financed by those of its core, such as Germany and France, as well as Benelux and Austria, although to a lesser extent, and eventually repaid in a large part by Italy too (i.e. 2010s European Stability Mechanism (ESM)), which had itself been incurring a trade deficit, nonetheless financed through domestic real tradable assets ta (i.e. accumulated domestic real savings $\pi \dot{s}$[4]).

As a consequence, potential domestic damage of sudden stops in foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$ does not only entail a default on extant foreign net real borrowing $rb - e \cdot r^* b^*$ but a relative loss in foreign investment repayment credibility as well and an attendant halt in domestic importation, with all of its effects on domestic living standards.

By way of example yet, if the American Dollar were to cease being the international reserve currency the USA would no longer run a trade deficit, by the Triffin dilemma, but if their trade deficit were to remain in place owing to another cause (e.g. trade policy, exchange rate policy, market demand) the USA, initially still short of domestic real tradable assets ta, could then face the risk of a sudden stop in foreign net real borrowing $nb \equiv -nl = \pi_E \dot{b} - \dot{e} \cdot \dot{b}^*$, especially from China.

In fact, the conspicuous volume of American Dollars in circulation as a result of their surmised demotion from international reserve currency status would trigger a correspondingly conspicuous real exchange rate depreciation of the American Dollar and thus hardly allow the USA to incur a trade deficit at all, all else notwithstanding.

The American trade surplus likely resulting therefrom, on the contrary, would then permit the USA to accumulate sufficient foreign reserves to finance a new trade deficit whenever the dormant causes for one may subsequently prevail anew or reemerge.

4. Triffin dilemma

4.1 Triffin paradox. The Triffin dilemma refers to the paradox whereby a country whose external currency is an international reserve currency, be it by decree or OMOs (e.g. Bretton Woods Agreement or the era thereafter), is bound to primarily exhibit a trade deficit and an attendant imbalance in foreign reserves and secondarily dilute its control over its (i) exchange rate policy and (ii) monetary policy if under internal financial openness and a single currency against which others are managed or fixed.

The reason for which a currency may be elected as one of international reserves seems to ultimately depend upon the short run growth (i.e. internal stability) and long run growth (i.e. wealth, external stability) of a country's economy, thereby owing to its consumption and taxation solidity.

[4] $\frac{\dot{p}\dot{s}}{p} = \pi \dot{s}$, where, $\forall p_I \in \mathbb{R}_{++},\ p = p_E + p_I$, that is, domestic prices p are the sum of domestic external prices p_E and positive domestic internal prices p_I.

Acting as a benchmark, a country also suffering from monetary policy dilution as a result of the Triffin dilemma cannot logically be one to which the Mundell trilemma strictly applies. Although it featured internal financial openness, real exchange rate fixation (including management) would at most only be suffered by such a country, not governed, and monetary policy in all events diluted, precisely because of other countries' fixation or management of their own external currencies against that of the one in question, at given international prices. Such a country would have in fact either way accepted the election of its external currency as one of international reserves.

4.2 Monetary policy dilution. The degree to which its monetary policy may resist dilution is thus related to the willingness of other countries to experience the effects of its monetary policy in its stead, in their respective external or internal currencies, depending on the currency regime they have elected.

A nominal monetary expansion or contraction by the international reserve currency country at given prices at home and abroad would be immediately neutralised by one abroad, on the foreign exchange market, effectively projecting the ratio target abroad of foreign real international reserves denominated in the international reserve currency to the international reserve currency's real monetary base on the foreign real exchange rate.

Following the Great Recession of the 2000s the USA's quantitative easing (QE) should have been consequently neutralised, on balance, resulting in the appropriation of their monetary policy abroad, owing to the American Dollar's status as an international reserve currency, especially on consideration of the more or less concomitant policies of monetary expansion abroad, in most of the of the Western world (e.g. Japan, United Kingdom (UK), EA) and in an effort to annul the global saving glut; however, the intensity of their QE, in the time frame of its effectiveness, resulting from the asymmetry of the global saving glut skewed towards the USA, probably led most other countries not to appropriate themselves of such an extreme approach to monetary policy there and then.

If the willingness of other countries to experience the effects of its monetary policy in its stead were firm then the country presenting the international reserve currency could elude monetary policy dilution by employing either (i) a double currency regime or (ii) a single currency regime, both enacted via internal financial closure.

4.3 International reserve currency alternatives. For such a reason too had John Maynard Keynes proposed the adoption of the Bancor currency for international trade and commerce. Future imbalances in the current accounts and in the capital accounts of countries due to the multinational real exchange rates passing through the Bancor would have been additionally and particularly neutralised by the International Clear Union (ICU), via transfers and forced realignments in multinational real exchange rates.

Despite the rejection of the Bancor project, in favour of the Bretton Woods Agreement (i.e. nominal exchange rate fixation, internal financial closure and American Dollar fixed convertibility into gold), the ICU's role was acquired by the International Monetary Fund (IMF), rather than the nominally apter World Bank, which instead took up the role of an international agent of financial aid, theoretically befitting the IMF to a higher degree.

The Bancor currency nonetheless regained attention in the recent past, owing to both the Triffin dilemma and the Rodrik trilemma, formalised by Saccal [16][5]. The IMF's special drawing rights (SDRs), the Venezuelan Petromoneda and the international currency system proposed by the Association of South East Asian Nations (ASEAN) have been particularly endorsed, in the especial capacity of a supra or transnational unit of account currency.

As a consequence, the Triffin dilemma is a dilemma in that such a country cannot simultaneously be (i) one by the international reserve currency status and (ii) one not featuring a trade deficit. A double currency regime, enacted via internal financial closure, would in fact be preferable for the avoidance of

[5] Saccal [16] formalised Rodrik's globalisation paradox by proving that growth mismatches in multilateral external real money supply, under trade, give rise to ones in multilateral external real output, whose lasting nature until natural autarky, due to Kaldorian external price endogeneity in external real output, can only be resolved through earlier trade cessation or multilateral external real money supply growth equalisation and not by balance of payments transfers. The mercantile dilemma resulting from such, the Saccal dilemma, is therefore the optimal selection between (i) artificial currency areas, whereby multilateral external real money supply growth equality under trade guarantees growth equality in multilateral external real output, and (ii) modern protectionism, whereby trade is simply interrupted, thereby conjointly eschewing globalisation, whereby multilateral external real output growth equality under trade results from a declining equalisation in multilateral external real money supply growth.

monetary policy dilution resulting from the Triffin dilemma as much as a single currency regime under internal and external financial closure: either would exhibit Pareto improvements over internal financial openness and a single currency against which others are managed or fixed.

Such a double currency regime would although be ultimately preferable for a more fluid regulation of import export incentives, as discussed above, through the nominal exchange rate between domestic external currency M_{SE} and domestic internal currency M_{SI}, provided it be not decreed. Under external financial openness a double currency regime by which the nominal exchange rate between domestic external currency M_{SE} and domestic internal currency M_{SI} is not decreed would moreover permit the domestic private sector to pursue investments abroad, by exchanging domestic external currency M_{SE} with foreign currency M^*_S.

The Triffin paradox appears to have been confirmed by contemporary data in that the American Dollar has been the international reserve currency since 1944 (i.e. Bretton Woods Agreement) and the USA, which had begun by exhibiting a trade surplus and the greatest volume of gold reserves in the world, being still in place, have tended to run a trade deficit until today[6], rather consistently, that is to say, matched by a de-cumulation of foreign reserves and exasperated by the Petrodollar recycling phenomenon[7].

The difference[8] between their foreign reserves and their indebtedness against the rest of the world, as a matter of fact, which speaks to their capital account, has featured negativity and whenever juxtaposed to that of other countries results as being the greatest worldwide. One's additional task can then be that to formalise the Triffin dilemma, both in terms of the current and capital account, to be sure.

4.4 Real capital account. The capital account is broadly defined as the sum of foreign direct investment abroad from home (i.e. non-securities investment[9], which is long term), foreign portfolio investment abroad from home (i.e. securities investment[10], which is short term), other investments abroad from home (i.e. loans and commercial bank flows, which are short term) and the reserve account (i.e. tradable assets and foreign aid at home from abroad).

Foreign portfolio investment and other investments are sometimes grouped under short term investment; alternatively, other investments can be recorded under the reserve account, which can itself be discretionally omitted and at times is (i.e. capital account narrowly defined).

Domestic real capital account ca_2 was therefrom defined as the sum of foreign net real lending nl, itself the difference between foreign real lending $\dot{e} \cdot \dot{b}^*$ and foreign real borrowing $\pi_E \dot{b}$, and domestic real tradable assets $ta:\ ca_2 = nl + ta = (\dot{e} \cdot \dot{b}^* - \pi_E \dot{b}) + ta$. Foreign net real lending nl measures foreign direct investment and short term investment abroad from home; domestic real tradable assets ta measures the reserve account.

A rise in foreign real lending $\dot{e} \cdot \dot{b}^*$ signifies a figurative outbound flow (i.e. credit) of domestic real tradable assets ta, being sources of funds for foreign importation, and a future domestic claim on them abroad (i.e. on foreign real tradable assets ta^*).

A rise in foreign real borrowing $\pi_E \dot{b}$ signifies a figurative inbound flow (i.e. debt) of foreign real tradable assets ta^*, being sources of funds for domestic importation, and a future foreign claim on them at home (i.e. on domestic real tradable assets ta).

An accumulation and de-cumulation of domestic real tradable assets ta or of foreign net real lending nl (which are also assets insofar as they be future domestic net claims on domestic real tradable assets ta abroad) thence respectively characterise a surplus and deficit in domestic real capital account ca_2 and are indicative of ones in domestic real current account ca; alternative accountancy conventions reverse the relationship, measuring foreign direct investment and short term investment at home from abroad and a negative reserve account (i.e. negative tradable assets and foreign aid abroad from home): $ca = ca_2$ such that $ca_{\gtrless 0} = ca_{2 \gtrless 0}$ becomes $ca = -ca_2$ such that $ca_{\gtrless 0} = -ca_{2 \lessgtr 0}$.

[6] https://data.oecd.org/trade/current-account-balance.htm, https://en.wikipedia.org/wiki/File:Cumulative_Current_Account_Balance.png, https://www.imf.org/external/datamapper/BCA_NGDPD@WEO/ISR

[7] The Petrodollar recycling phenomenon describes the dynamics of non-American (developing) crude oil exporters reinvesting their petroleum sale proceeds denominated in American Dollars abroad, especially in the USA and especially in American public bonds: $s < i \longleftrightarrow ca < 0$ and $s- \uparrow i =\downarrow ca \vdash s \ll i \longleftrightarrow ca \ll 0$.

[8] https://en.wikipedia.org/wiki/File:Country_foreign_exchange_reserves_minus_external_debt.png

[9] Firm capital (sole trader, partnership, private and public limited liability).

[10] Firm shares (private and public limited liability), private and public bonds (financial intermediaries and government), derivatives and other securities (currency, funds etc.).

Granted that the current account record a country's international transactions of goods and services (i.e. commodities) and that the capital account record those pertaining to assets, the distinction of the financial account from the capital account is such that the financial account records a country's international, market asset transactions and that the capital account records a country's international, non-market asset transactions.

4.5 Real balance of payments. Depending on domestic real financial account fa's presence, domestic real balance of payments *bop* is such that either (i) domestic real current account ca must equal domestic real capital account ca_2 or (ii) the sum of domestic real current account ca and domestic real capital account ca_2 must equal domestic real financial account fa, often through a statistical discrepancy term or corrector (i.e. balancing item) in either case; formally: (i) *bop* such that, $\forall \epsilon \in \mathbb{R}_+,\ ca \pm \epsilon = ca_2 \vdash bop = ca - ca_2 \pm \epsilon = 0$; (ii) *bop* such that, $\forall \epsilon \in \mathbb{R}_+,\ ca + ca_2 \pm \epsilon = fa \vdash bop = (ca + ca_2) - fa \pm \epsilon = 0$.

Such equalities are reflective of the fact that hypothetical discrepancies between exportation earnings and importation expenses, at a multinational level, must be offset by ones in tradable assets or promises to repay the difference at bargained rates of interest in the future, at the same level: absent tradable assets abroad, a current account surplus at home is financed through foreign net lending abroad from home; absent tradable assets at home, a current account deficit at home is financed through foreign net borrowing at home from abroad.

Let one then conceive of domestic real balance of payments *bop* as fundamentally characterised by domestic real current account ca and domestic real capital account ca_2 alone, eschewing the definition of domestic real financial account $fa :\ bop$ such that $ca = nx + ny_r + ntr_r = nl + ta = ca_2$.

Across a country's balance of payments receipts from abroad are recorded as credit entries and payments abroad are recorded as debit ones; formally: *bop* such that $\overset{c}{ca} = \overset{c}{nx} + \overset{c}{ny_r} + \overset{c}{ntr_r} = (\overset{c}{ex} - e \cdot \overset{d}{im}) + (\overset{c}{y_r} - \overset{d}{y_p}) + (\overset{c}{tr_r} - \overset{d}{tr_p}) = (\dot{e} \cdot \overset{c}{\dot{b}^*} - \pi_E \overset{d}{\dot{b}}) + \overset{c}{ta} = \overset{c}{nl} + \overset{c}{ta} = \overset{c}{ca_2}$.

In domestic real current account ca domestic real exports ex, domestic real output receipts y_r and domestic real transfer receipts tr_r represent revenue receipts, being income, just as domestic imports real money value $e \cdot im$ represents a cost payment, being spending.

Correspondingly, in domestic real capital account ca_2 foreign real lending $\dot{e} \cdot \dot{b}^*$ and domestic real tradable assets ta represent asset acquisitions, being domestic claims on real tradable assets abroad and possessions thereof at home, just as foreign real borrowing $\pi_E \dot{b}$ represents an asset cession, being foreign claims on real tradable assets at home.

Extant foreign real borrowing weighted at domestic real interest rate rb, being the restitution abroad of the borrowed real principal sum at its real interest rate, is instead recorded in domestic real current account ca under domestic real transfer payments $tr_p :\ tr_p = \sum_{j=1}^{n} s_j$, where summand $s_j = rb$ and summand $s_{\neg j} \in \mathbb{R}_+$, that is, $rb + \ldots = tr_p$.

Foreign real lending weighted at foreign real interest rate and domestic real exchange rate $e \cdot r^* b^*$, being the restitution at home of the lent real principal sum at its real interest rate and real exchange rate, is accordingly recorded in domestic real current account ca under domestic real transfer receipts $tr_r :\ tr_r = \sum_{j=1}^{n} s_j$, where summand $s_j = e \cdot r^* b^*$ and summand $s_{\neg j} \in \mathbb{R}_+$, that is, $e \cdot r^* b^* + \ldots = tr_r$.

Consequently, an open economy's national accounting identity cannot be written to embed its entire balance of payments, but its current account alone, lest its external sector be shunned altogether; formally: $y \neq c + g + i + bop = c + g + i + (ca - ca_2 \pm \epsilon) = c + g + i + [(ex - e \cdot im + y_r - y_p + tr_r - tr_p) - (\dot{e} \cdot \dot{b}^* - \pi_E \dot{b} + ta) \pm \epsilon] = c + g + i$, but $y = c + g + i + ca$.

4.6 Real balance of payments imbalances. Domestic real balance of payments surplus or balance (i.e. non-deficit) $bop_{\geq 0}$ is thence defined as the difference between domestic real current account surplus $ca_{>0}$ and domestic real capital account net of domestic real tradable assets, foreign net real lending, balance or surplus (i.e. non-deficit) $(ca_2 - ta)_{\geq 0} = nl_{\geq 0}$; domestic real balance of payments deficit or balance (i.e. non-surplus) $bop_{\leq 0}$ is specularly defined as the difference between domestic real current account deficit $ca_{<0}$ and domestic real capital account net of domestic real tradable assets, foreign net real lending, balance or deficit (i.e. non-surplus) $(ca_2 - ta)_{\leq 0} = nl_{\leq 0} :\ bop_{\gtreqless 0}$ such that $ca_{\gtrless 0} - (ca_2 - ta)_{\gtreqless 0} = \underbrace{(nx + ny_r + ntr_r)}_{\gtrless 0} - \underbrace{nl}_{\gtreqless 0} \gtreqless 0$.

Domestic real balance of payments surplus and deficit $bop_{>0}$ and $bop_{<0}$ are respectively met with a rise and a fall in domestic real tradable assets $ta:$ $bop_{>0}$ such that $\uparrow_2 ca- \uparrow nl =\uparrow ta \vdash\uparrow ca - nl =\uparrow ta$; $bop_{<0}$ such that $\downarrow_2 ca- \downarrow nl =\downarrow ta \vdash\downarrow ca - nl =\downarrow ta$.

Domestic real balance of payments balance $bop_{=0}$ is met with a rise or a fall in foreign net real lending $nl:$ $bop_{=0}$ such that $\uparrow ca- \uparrow nl =\uparrow\downarrow ta = ta$ or $\downarrow ca- \downarrow nl =\downarrow\uparrow ta = ta$.

In detail, domestic real current account surplus $ca_{>0}$ signifies an inflow of domestic real tradable assets ta (e.g. foreign reserves, precious metals[11], demand and supply inelastic securities) or an outflow of foreign net real lending nl; specularly, domestic real current account deficit $ca_{<0}$ signifies an outflow of domestic real tradable assets ta or an inflow of foreign net real borrowing $nb \equiv -nl$. One emphasises that the "or" conjunction is inclusive, both times.

Proposition 4.7 (Triffin paradox) *If the domestic external currency is an international reserve currency then (i) the domestic country faces a trade deficit and (ii) extant foreign net real borrowing exceeds domestic real tradable assets. Formally:*

$$M_{SE} \sim RC \longrightarrow (i)\ ex < e \cdot im,\ (ii)\ rb - e \cdot r^*b^* > ta.$$

Proof. Lemma 4.7.1 (First consequent) Direct proof. $M_{SE} \sim RC \vdash (\downarrow x \in\downarrow M_{SE}) = (\uparrow x^* \in\uparrow M^*_S)$, by $M_{SE} \sim M^*_S \vdash\downarrow_2\uparrow nx =\downarrow ex- \downarrow e\cdot \uparrow im \vdash\downarrow nx$ such that $nx < 0 \vdash ex < e \cdot im$.

Positive $nm \equiv -nx$, coupled with $rb - e \cdot r^*b^*$, are financed by $\pi_E\dot{b} - \dot{e} \cdot \dot{b}^*,\ ta$ and $ny_r + ntr_r$: $(\pi_E\dot{b}-\dot{e}\cdot\dot{b}^*)+ex-ta+(ny_r+ntr_r) \geq e\cdot im+(rb-e\cdot r^*b^*) \vdash (\pi_E\dot{b}-\dot{e}\cdot\dot{b}^*)+(e\cdot r^*b^*-rb)-ta+(ny_r+ntr_r) \geq e \cdot im - ex = nm = -nx$, where $e \cdot im > ex$.

Lemma 4.7.2 (Second consequent) Direct proof. $\downarrow M_{SE} \vdash\downarrow p_E \vdash\uparrow m_{SE} \vdash\uparrow_2\downarrow nx =\uparrow ex- \uparrow e\cdot \downarrow im \vdash\uparrow nx$ such that $nx > 0 \vdash ex > e \cdot im$, but $(\downarrow x \in\downarrow M_{SE}) = (\uparrow x^* \in\uparrow M^*_S)$, by $M_{SE} \sim M^*_S$, since $m_{SE} \sim RC$, whence $nx < 0 \vdash ex < e \cdot im$ anew.

It follows that, for $n\bar{y}_r$ and $n_2\bar{t}r_r$, $\uparrow (\pi_E\dot{b} - \dot{e} \cdot \dot{b}^*)+ \downarrow (e \cdot r^*b^* - rb)- \downarrow ta$ until $\pi_E\dot{b} - \dot{e} \cdot \dot{b}^* \gg 0,\ e \cdot r^*b^* - rb \ll 0$ and $ta \approx 0$; specifically, $rb - e \cdot r^*b^* > ta \vdash rb - e \cdot r^*b^* - ta > 0$.

Now, if $(\downarrow x^* \in\downarrow M^*_S) = (\uparrow x \in\uparrow M_{SE})$, by $M^*_S \sim M_{SE}$, such that $\uparrow ta \vdash rb - e \cdot r^*b^* \leq ta$ it follows that $\uparrow_2\downarrow nx =\uparrow ex- \uparrow e\cdot \downarrow im \vdash\uparrow nx$ such that $nx > 0 \vdash ex > e\cdot im$, whereby $\downarrow (\pi_E\dot{b}-\dot{e}\cdot\dot{b}^*)+ \uparrow (e\cdot r^*b^*-rb)- \uparrow ta$, for $n\bar{y}_r$ and $n_2\bar{t}r_r$, until $\pi_E\dot{b} - \dot{e} \cdot \dot{b}^* \ll 0,\ e \cdot r^*b^* - rb \gg 0$ and $ta \gg 0$.

However, $m_{SE} \sim RC \vdash (\downarrow x \in\downarrow M_{SE}) = (\uparrow x^* \in\uparrow M^*_S)$, by $M_{SE} \sim M^*_S \vdash nx < 0 \vdash ex < e \cdot im$ and $rb - e \cdot r^*b^* > ta \vdash rb - e \cdot r^*b^* - ta > 0$ anew.

Since $\tilde{n}(M_{SE}) = e \cdot \tilde{n}(M^*_S) \longrightarrow \tilde{n}(M^*_S) = e^{-1}\tilde{n}(M_{SE}) = e^*\tilde{n}(M_{SE})$ we note that $rb - e \cdot r^*b^* - ta = n_{rb}(M_{SE})-e\cdot n_{r^*b^*}(M^*_S)-n_{ta}(M_{SE}) = n_{rb}(M_{SE})-n_{r^*b^*}(M_{SE})-n_{ta}(M_{SE}) = e\cdot[n_{rb}(M^*_S)-n_{r^*b^*}(M^*_S)-n_{ta}(M^*_S)]$. *QED*

It must be stressed that, in the proof of the second consequent, the readjustment in domestic external prices p_E pertinent to the fall in domestic external currency M_{SE}, raising domestic external real money supply m_{SE} and depreciating domestic real exchange rate e, is punctually offset by another fall in domestic external currency M_{SE}, due to its international reserve currency status valued in real terms.

4.8 International currency status changes. The promotion of an external currency to one of international reserves implies a decrease of supply in such a currency, just as its demotion from such a status implies an increase of supply in it; the converse, however, does not hold, for a change in an external currency supply need not be driven by its promotion to or demotion from the status of an international reserve currency, but by other dynamics (e.g. exchange rate policy, alternative market demand): $M_{SE} \sim RC \not\dashv\vdash (\downarrow x \in\downarrow M_{SE})$ and $(M_{SE} \not\sim RC$, given $M_{SE} \sim RC) \not\dashv\vdash (\uparrow x \in\uparrow M_{SE})$.

It follows that while a country featuring a trade deficit need not feature an external currency as one of international reserves, but incur the trade deficit owing to other causes (e.g. trade policy, exchange rate policy, market demand), one can state that if the domestic external currency is no longer an international reserve currency (i.e. international reserve currency status demotion) then the domestic country no longer faces a trade deficit, but not *vice versa:* $(M_{SE} \not\sim RC$, given $M_{SE} \sim RC) \longrightarrow (ex \geq e\cdot im$, given $ex < e\cdot im)$,

[11] Until the early 1980s in particular, when the Bank of Italy famously acquired independence of the Italian treasury and the country entered the European Monetary System (ESM), reminiscent of the Latin Monetary Union (LMU), Italy accumulated ample gold reserves precisely through its exportations, yet being the third country in the world in terms of gold reserve tonnes (see https://www.gold.org/goldhub/data/gold-reserves-by-country).

since the antecedent implies $(\uparrow x \in \uparrow M_{SE}) = (\downarrow x \in \downarrow M_S^*)$, by $M_{SE} \sim M_S^*$, whence $\uparrow ex \geq \uparrow e \cdot \downarrow im$[12] such that (i) $\uparrow_2 \downarrow nx \vdash \uparrow nx$ and $nx > 0 \vdash ex > e \cdot im$ or, at equal changes but a different $ex < e \cdot im$ value, (ii) $\bar{nx}$ and $nx = 0 \vdash ex = e \cdot im$; $(ex \geq e \cdot im$, given $ex < e \cdot im) \not\longrightarrow (M_{SE} \not\sim RC$, given $M_{SE} \sim RC)$, just as $ex < e \cdot im \not\longrightarrow M_{SE} \sim RC$, in view of due counterexamples.

4.9 Domestic real private and public consumption changes. It is argued that the domestic trade deficit pertaining to the promotion of the domestic external currency to an international reserve currency may act as an incentive to incur greater domestic real private and public consumption $c + g$, especially public, in view of the shortage of domestic real savings s in relation to domestic real firm consumption i : $M_{SE} \sim RC \vdash nx < 0$ and, for constant ny_r and ntr_r, $s < i \longleftrightarrow ca < 0$, thus, it is hypothesised that $\downarrow s = y - \uparrow (c + g) \vdash s \ll i \longleftrightarrow ca \ll 0$.

The increase in domestic real private and public consumption $c + g$ ultimately results in an offsetting rise in domestic real output y, through domestic external or internal money demand m_{DE} or m_{DI}, domestic real output y being the sum of domestic external real output y_E and domestic non-tradable real output y_{NTR}; if the increase in domestic real private and public consumption $c + g$ originated from domestic external money demand m_{DE}, however, domestic real exchange rate e would be affected too, undergoing an appreciation and thereby effectively causing a further domestic trade deficit: $y_{TR} + y_{NTR} = (y_E - nx) + y_{NTR} = c + g + i + ny_r + ntr_r \vdash y_E + y_{NTR} = y = c + g + i + (nx + ny_r + ntr_r) = c + g + i + ca$; $y_{NTR} = f(\overset{+}{m}_{SI},\ \overset{+}{m}_{DI})$, domestic internal real money supply $m_{SI} = \frac{M_{SI}}{p_I} \left(= \frac{M_{SE}}{p_I}\right)$[13] and, for domestic autonomous external and internal money demand am_{DE}, $am_{DI} \in \mathbb{R}_{++}$, $m_{DE} = f(\overset{+}{am}_{DE})$ now and domestic internal money demand $m_{DI} = f(\overset{+}{am}_{DI})$; additionally, domestic real private and public consumption $(c + g) \equiv \hat{c} = f(\overset{+}{m}_{DE},\ \overset{+}{m}_{DI})$, absent loss of generality; thus, (i) if $y_{y_{NTR}} y_{NTR_{m_{DI}}} m_{DI_{am_{DI}}}$, reflective of $\hat{c}_{m_{DI}} m_{DI_{am_{DI}}}$, then $\uparrow y = \uparrow (c + g) + i + ca \vdash s = \uparrow y - \uparrow (c + g) = i + ca$, whence $s \not\ll i \longleftrightarrow ca \not\ll 0$, and (ii) if $y_{y_E} y_{E_{m_{DE}}} m_{DE_{am_{DE}}}$, reflective of $\hat{c}_{m_{DE}} m_{DE_{am_{DE}}}$, or $y_{y_{NTR}} y_{NTR_{m_{DI}}} m_{DI_{am_{DI}}}$ (which is neutral), reflective of $\hat{c}_{m_{DI}} m_{DI_{am_{DI}}}$ (which is neutral), then $e_{m_{DE}} m_{DE_{am_{DE}}}$, $ex_e e_{m_{DE}} m_{DE_{am_{DE}}}$ and $im_e e_{m_{DE}} m_{DE_{am_{DE}}}$ such that $s = \uparrow y - \uparrow (c + g) = \uparrow i + \downarrow ca \vdash \uparrow y = \uparrow (c + g) + \uparrow i + \downarrow ca$, whence $s \ll i \longleftrightarrow ca \ll 0$.

For completeness, the rationale in favour of anti-cyclical fiscal policy in view of internal temporary demand shocks is such that a fall in domestic real private consumption c driven by one in domestic internal money demand m_{DI} exceeds the attendant fall in domestic real output y, passing through domestic non-tradable real output y_{NTR}, whereby domestic real savings s exceed the sum of domestic real firm consumption i and domestic real current account ca; in a simpler form: $\forall \varepsilon_c,\ \varepsilon_g \in \mathbb{R}_{++}$, being white noises[14] in private and public consumption respectively, $y = f(\overset{+}{c},\ \overset{+}{g},\ \overset{+}{i},\ \overset{+}{ca})$, $c = f(\overset{+}{\varepsilon}_c)$ and $g = f(\overset{+}{\varepsilon}_g)$, absent loss of generality, $-c_{\varepsilon_c} < -y_c c_{\varepsilon_c}$ is such that $s = \downarrow y - \downarrow c - g > i + ca$, whereby $s = \downarrow y - \downarrow c - \uparrow g = i + ca$, through $y_g g_{\varepsilon g}$.

Accordingly, a fall in domestic real private consumption c driven by one in domestic external money demand m_{DE} exceeds the attendant fall in domestic real output y, passing through domestic non-tradable real output y_{NTR}; a fall in domestic real private consumption c driven by one in domestic external money demand m_{DE} nevertheless also passes through domestic external real output y_E, causing it to increase together with domestic real current account ca, on account of a depreciation in domestic real exchange rate e, so that the ultimate effect on domestic real output y and domestic real savings s thereby be uncertain; in a simpler form: $\forall e = f(\overset{-}{c})$, absent loss of generality, $-c_{\varepsilon_c} < -y_c c_{\varepsilon_c}$ and $-y_{ca} ca_e e_c c_{\varepsilon_c} - y_c c_{\varepsilon_c} \gtreqless 0 \vdash y_{ca} ca_e e_c c_{\varepsilon_c} + y_c c_{\varepsilon_c} \gtreqless 0$ are such that $s = \uparrow\downarrow y - \downarrow c - g \gtreqless i + \uparrow ca$, *ceteris paribus.*

5. SDPE model

5.1 Optimisation problem. To make greater sense of the importation finances and of the Triffin dilemma hitherto analysed one is to develop an SDPE model and assess the resulting dynamical system, by

[12]Specifically, $ex_e e_{r_E} r_{E_{m_{SE}}} \geq e_{r_E} r_{E_{m_{SE}}} \cdot im_e e_{r_E} r_{E_{m_{SE}}}$.
[13]One observes that $m_{SI} = \frac{M_{SI}}{p_I} = \frac{M_{SE}}{p_I} = \frac{M_S}{p_I}$.
[14]Generic white noise is normally distributed with a 0 mean and a finite variance: $\varepsilon \sim \mathcal{N}(0,\ \sigma^2)$.

means of stability analysis. Let one firstly specify that all treated variables have ultimately been a function of non-negative time t. Consider then the following non-linear programming problem:

$$V[b(0),\ 0] = \max_{\{b(t),\ ex(t)\}_{t\in\mathbb{R}_+}} \int_0^\infty u[b(t),\ ex(t),\ t]dt =$$
$$= V[b(0)] = \max_{\{b(t),\ ex(t)\}_{t\in\mathbb{R}_+}} \int_0^\infty e^{-\rho t}u[b(t),\ ex(t)]dt =$$
$$= \max_{\{b(t),\ ex(t)\}_{t\in\mathbb{R}_+}} \int_0^\infty e^{-\rho t}ln[ex(t)]dt\ s.t.$$
$$\dot{b}(t) = v[b(t),\ ex(t)] = \pi_E^{-1}\{[e\cdot im - ex(t)] + [rb(t) - e\cdot r^*b^*] + ta - [ny_r + n_2tr_r] + \dot{e}\cdot\dot{b}^*\},$$

where initial extant foreign real borrowing $b(0) = b_0,\ \forall f : \mathbb{R}_+ \to Y$, being a suitably given function, $\lim_{t\to\infty} f(t)b(t) = b(t) \geq b_1$, extant foreign real borrowing $b(t) \in B \subseteq \mathbb{R}$ and domestic real exports $ex(t) \in EX \subseteq \mathbb{R}$. It follows that terms $\pi_E,\ e,\ im,\ r,\ r^*,\ b^*,\ ta,\ ny_r,\ n_2tr_r,\ \dot{e}$ and $\dot{b}^*$ are heterogeneous components of the ordinary differential equation (ODE) for extant foreign real borrowing weighted at domestic real interest rate $rb(t)$ and are thereby constant.

In fact, domestic external inflation π_E and domestic real interest rate r can be already determined as respectively being a positive and a non-negative parameter, together with domestic negative discount rate ρ lying in an open, real interval between 0 and 1 : $\pi_E \in \mathbb{R}_{++},\ r \in \mathbb{R}_+$ and $\rho \in (0,\ 1) \subset \mathbb{R}_{++}$.

The optimisation problem is not one of a representative agent, but of a country's domestic real trade balance as a whole, thereby eluding the shortcomings highlighted by the "Anything goes"[15] theorem.

Present value[16] of summed instantaneous changes in the domestic country's trade balance utility function $\int_0^\infty e^{-\rho t}dU[b(t),\ ex(t)] = \int_0^\infty e^{-\rho t}u[b(t),\ ex(t)]dt$ is such that instantaneous time t derivative $u[b(t),\ ex(t)]$ of said utility function $U[b(t),\ ex(t)]$ equals natural logarithm $ln[\cdot]$ of domestic real exports $ex(t)$, being the control variable: $\int_0^\infty e^{-\rho t}dU[b(t),\ ex(t)] = \int_0^\infty e^{-\rho t}u[b(t),\ ex(t)]dt = \int_0^\infty e^{-\rho t}ln[ex(t)]dt$, where $u[b(t),\ ex(t)] = (dt)^{-1}dU[b(t),\ ex(t)]$ is, as stated, domestic country's trade balance utility function $U[b(t),\ ex(t)]$ derived at instantaneous time t. Such a functional specification is a subclass of the constant relative risk aversion (CRRA) utility function[17].

5.2 Hamiltonians and optimal solutions. Consequently, the present value Hamiltonian equation[18] of the optimisation problem, wherein summed instantaneous changes in the domestic country's trade balance utility function $\int_0^\infty dU[b(t),\ ex(t)] = \int_0^\infty u[b(t),\ ex(t)]dt$ are omitted in favour of domestic country's trade balance derived utility function $u[b(t),\ ex(t)]$ absent loss of generality[19], is $\mathcal{H}_P[b(t),\ ex(t),\ \lambda(t),\ t] = e^{-\rho t}u[b(t),\ ex(t)] + \lambda(t)v[b(t),\ ex(t)] = e^{-\rho t}\{ln[ex(t)] + \mu(t)\dot{b}(t)\}$, since future value Hamiltonian co-state multiplier $\mu(t) = e^{\rho t}\lambda(t)$, where $\lambda(t) = e^{-\rho t}\mu(t)$ is the present value Hamiltonian co-state multiplier. The future value Hamiltonian equation of the optimisation problem is therefore $\mathcal{H}_F[b(t),\ ex(t),\ \mu(t)] = ln[ex(t)]+$

[15] https://en.wikipedia.org/wiki/Sonnenschein–Mantel–Debreu_theorem

[16] It is obtained as follows: future or current value $FV = \lim_{s\to\infty} PV\left(1+\frac{\rho}{s}\right)^{\left(\frac{s}{\rho}\right)\rho t} = \lim_{y\to\infty} PV\left(1+\frac{1}{y}\right)^{y\rho t} = PVe^{\rho t}$, whence present or constant value $PV = e^{-\rho t}FV$, where s is a positive (infinitesimal) scaling factor and future value FV and present value PV are real numbers. Furthermore, ρ is typically termed "discount rate", despite being an interest on present value, and $-\rho$ is accordingly termed "discount factor". We prefer terming $-\rho$ "discount rate", being a discount on future value, and ρ "negative discount rate" thereby.

[17] https://en.wikipedia.org/wiki/Isoelastic_utility

[18] https://en.wikipedia.org/wiki/Hamiltonian_(control_theory)

[19] The omission of summed instantaneous changes in the domestic country's trade balance utility function $\int_0^\infty dU[b(t),\ ex(t)] = \int_0^\infty u[b(t),\ ex(t)]dt$ without loss of generality from the present value Hamiltonian equation of the optimisation problem in question is due to the mathematical derivation of the Hamiltonian equation of a generic optimisation problem. For such a reason, one could argue, is the model deemed static, as opposed to dynamic, although the more general consideration of summed instantaneous changes in the domestic country's trade balance utility function $\int_0^\infty dU[b(t),\ ex(t)] = \int_0^\infty u[b(t),\ ex(t)]dt$ may justify the "dynamic" appellative instead. In fact, even though summed instantaneous changes in a function may initially seem different from the individual instantaneous changes in the function themselves, doubtless being so computationally, owing to how real numbers are treated, it is not the logically valid (and sound) case, for in continuous time all dynamics are conceptually static, owing to its uncountability and pertinent absence of past, present and future, the two adjectives being as it were equivalent and allowing one to conclusively opt for the latter.

$\mu(t)\dot{b}(t) = e^{\rho t}\mathcal{H}_P[b(t),\ ex(t),\ \lambda(t),\ t] = e^{\rho t}\{e^{-\rho t}ln[ex(t)] + \lambda(t)\dot{b}(t)\} = e^{(\rho-\rho)t}\{ln[ex(t)] + \mu(t)\dot{b}(t)\}$.

Necessary conditions for optimal solution $[\hat{ex}(t),\ \hat{b}(t)]$ should be[20]: $\forall t \in \mathbb{R}_+$, (i) $\mathcal{H}_{F_{ex(t)}}[b(t),\ ex(t),\ \mu(t)] = 0$, (ii) $\mathcal{H}_{F_{b(t)}}[b(t),\ ex(t),\ \mu(t)] = \rho\mu(t) - \dot{\mu}(t) = \rho[e^{\rho t}\lambda(t)] - \rho e^{\rho t}\lambda(t) = 0$, (iii) $\mathcal{H}_{F_{\mu(t)}}[b(t),\ ex(t),\ \mu(t)] = \dot{b}(t) = v[b(t),\ ex(t)]$, where $b(0) = b_0$ and $\lim_{t\to\infty} b(t) \geq b_1$, and (iv) $\lim_{t\to\infty} e^{-\rho t}\mathcal{H}_F[b(t),\ ex(t),\ \mu(t)] = \lim_{t\to\infty}\mathcal{H}_P[b(t),\ ex(t),\ \lambda(t),\ t] = \lim_{t\to\infty} e^{-\rho t}\mu(t)\hat{b}(t) = \lim_{t\to\infty}\lambda(t)\hat{b}(t) = 0$.

Such conditions are to accordingly become sufficient provided (i) the objective function and the constraint be at least twice continuously differentiable and weakly monotonic and (ii) the present value Hamiltonian equation be jointly concave in the state and control variables: (i-a) $\forall n \geq 2,\ u[b(t),\ ex(t)],\ v[b(t),\ ex(t)] \in C^n$; (i-b) $u[b(\tilde{t}),\ ex(\tilde{t})],\ b(\tilde{t}) \gtreqless u[b(t),\ ex(t)],\ b(t)$, if $\tilde{t} \gtreqless t$; (ii) $\mathcal{H}_P[b(t),\ ex(t),\ \lambda(t),\ t]$ is such that $H_{b(t)b(t)},\ H_{ex(t)ex(t)} \leq 0$ and $H_{b(t)b(t)}H_{ex(t)ex(t)} - H^2_{b(t)ex(t)} \geq 0$, where element $H_{b(t)ex(t)} = \frac{\partial^2\mathcal{H}_P[b(t),\ ex(t),\ \lambda(t),\ t]}{\partial b(t)\partial ex(t)}$ in a 2×2 Hessian matrix H.

The first order conditions (FOCs) of the optimisation problem thus are: (i) $\mathcal{H}_{F_{ex(t)}}[b(t),\ ex(t),\ \mu(t)] = u'[ex(t)] - \pi_E^{-1}\mu(t) = 0 \longrightarrow u'[ex(t)] = ex^{-1}(t) = \pi_E^{-1}\mu(t)$; (ii) $\mathcal{H}_{F_{b(t)}}[b(t),\ ex(t),\ \mu(t)] = \pi_E^{-1}r\mu(t) = \rho\mu(t) - \dot{\mu}(t) \longrightarrow \dot{\mu}(t) = \rho\mu(t) - \pi_E^{-1}r\mu(t) = \mu(t)(\rho - \pi_E^{-1}r)$; (iii) $\mathcal{H}_{F_{\mu(t)}}[b(t),\ ex(t),\ \mu(t)] = \dot{b}(t) = \pi_E^{-1}\{[e\cdot im - ex(t)] + [rb(t) - e\cdot r^*b^*] + ta - [ny_r + n_2tr_r] + \dot{e}\cdot\dot{b}^*\} = \pi_E^{-1}[-ex(t) + rb(t) + \kappa]$.

Thence ensues: $\mu(t) = \pi_E u'[ex(t)] = \pi_E ex^{-1}(t) \longrightarrow \dot{\mu}(t) = \pi_E u''[ex(t)]\dot{ex}(t) = \pi_E[(-1)ex^{-2}(t)](1) = -\pi_E ex^{-2}(t)\dot{ex}(t) = -\pi_E ex^{-2}(t)$ such that $\dot{\mu}(t) = \mu(t)(\rho - \pi_E^{-1}r) = -\pi_E ex^{-2}(t)\dot{ex}(t) = \pi_E ex^{-1}(t)(\rho - \pi_E^{-1}r) \longrightarrow \dot{ex}(t) = -ex(t)(\rho - \pi_E^{-1}r) = ex(t)(\pi_E^{-1}r - \rho)$, which alongside $\dot{b}(t) = \pi_E^{-1}[-ex(t) + rb(t) + \kappa]$ gives rise to a bi-dimensional, heterogenous system[21] of ODEs.

5.3 System solution. The bi-dimensional, heterogeneous system of ODEs is written thus:

$$\begin{bmatrix} \dot{ex}(t) \\ \dot{b}(t) \end{bmatrix} = \begin{bmatrix} \pi_E^{-1}r - \rho & 0 \\ -\pi_E^{-1} & \pi_E^{-1}r \end{bmatrix} \begin{bmatrix} ex(t) \\ b(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \pi_E^{-1}\kappa \end{bmatrix} \longleftrightarrow$$
$$\longleftrightarrow \dot{x}(t) = A(\Theta)x(t) + B(\Theta),$$

where parameter set $\Theta = \{\pi_E,\ r,\ \rho\} \subset \mathbb{R}_+$ as specified above. Now, the homogenous component of such a system is $\dot{x}(t) = A(\Theta)x(t)$, from which stability analysis can be conducted in terms of matrix $A(\Theta)$'s characteristic polynomial[22]: $A(\Theta) \equiv A = [(a_{11}\ a_{12})\ (a_{21}\ a_{22})]^\top$ such that matrix A's characteristic polynomial[23] $A(\lambda) = (A - \lambda I) = [(a_{11} - \lambda\ 0)\ (a_{21}\ a_{22} - \lambda)]^\top$, whereby $det[A(\lambda)] = (a_{11} - \lambda)(a_{22} - \lambda) - (0)a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} = 0 \longrightarrow \lambda_{1,\ 2} = \frac{(a_{11}+a_{22})\pm\sqrt{(a_{11}+a_{22})^2 - 4a_{11}a_{22}}}{2} = \frac{(a_{11}+a_{22})\pm\sqrt{a_{11}^2 + a_{22}^2 - 2a_{11}a_{22}}}{2} = \frac{(a_{11}+a_{22})\pm\sqrt{(a_{11}-a_{22})^2}}{2} = \frac{(a_{11}+a_{22})\pm(a_{11}-a_{22})}{2} = \frac{2a_{11}}{2},\ \frac{2a_{22}}{2} = a_{11},\ a_{22} = \pi_E^{-1}r - \rho,\ \pi_E^{-1}r$.

The corresponding eigenvectors $v_{1,\ 2}$ in $A(\lambda_{1,\ 2})v_{1,\ 2} = (A - \lambda_{1,\ 2}I)v_{1,\ 2} = 0 \longrightarrow Av_{1,\ 2} = \lambda_{1,\ 2}v_{1,\ 2}$ are computed thus:

[20] https://www.sas.upenn.edu/ jesusfv/lecturetechnical4_optimization_continuous.pdf
[21] https://en.wikipedia.org/wiki/Ordinary_differential_equation
[22] https://en.wikipedia.org/wiki/Characteristic_polynomial
[23] Scalar λ and eigenvalues $\lambda_{1,\ 2}$ originating therefrom are not to be confused with present value Hamiltonian co-state multiplier $\lambda(t)$.

$$A(\lambda_1)v_1 = \begin{bmatrix} a_{11}-\lambda_1 & 0 \\ a_{21} & a_{22}-\lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$$
$$\longrightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -\pi_E^{-1} & \rho & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -\pi_E\rho & 0 \end{array}\right], \text{ where } \begin{array}{c} \tilde{r}_1 = r_1 \\ \tilde{r}_2 = -\pi_E r_2 \end{array} \longrightarrow$$
$$\longrightarrow (1)v_{11} + (-\pi_E\rho)v_{12} = 0 \longrightarrow v_{11} = \pi_E\rho v_{12},\ v_{12} = v_{12} \longrightarrow$$
$$\longrightarrow v_{11} = \pi_E\rho, \text{ if } v_{12} = 1 \longrightarrow v_1 = \begin{bmatrix} \pi_E\rho \\ 1 \end{bmatrix};$$
$$A(\lambda_2)v_2 = \begin{bmatrix} a_{11}-\lambda_2 & 0 \\ a_{21} & a_{22}-\lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$$
$$\longrightarrow \left[\begin{array}{cc|c} -\rho & 0 & 0 \\ -\pi_E^{-1} & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \text{ where } \begin{array}{c} \tilde{r}_1 = -\rho^{-1} r_1 \\ \tilde{r}_2 = r_2 + \pi_E^{-1}\tilde{r}_1 \end{array} \longrightarrow$$
$$\longrightarrow (1)v_{21} + (0)v_{22} = 0 \longrightarrow v_{21} = -(0)v_{22},\ v_{22} = v_{22} \longrightarrow$$
$$\longrightarrow v_{21} = 0, \text{ if } v_{22} = 1 \longrightarrow v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The complementary solution of the system in question is consequently expressed as follows:

$$x_c(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 =$$
$$= C_1 e^{(\pi_E^{-1} r - \rho)t} \begin{bmatrix} \pi_E\rho \\ 1 \end{bmatrix} + C_2 e^{\pi_E^{-1} rt} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where solution coefficients $C_1,\ C_2 \in \mathbb{R}$. The particular solution of the system in question is analogously $x_p = K$, since the imposition of $x(t) = K$ in $\dot{x}(t) = Ax(t) + B$, where $B(\Theta) \equiv B$, yields $\dot{K} = 0 = AK + B \longrightarrow K = -A^{-1}B = [0\ \ -r^{-1}\kappa]^\top$, granted domestic real interest rate $r \in \mathbb{R}_{++}$ and heterogeneous component $\kappa \in \mathbb{R}$.

In the absence of initial conditions $ex(0)$ and $b(0)$, by which solution coefficients C_1 and C_2 remain unknown, the general solution of the bi-dimensional, heterogeneous system under scrutiny is therefore

$$x(t) = x_c(t) + x_p = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + K =$$
$$= C_1 e^{(\pi_E^{-1} r - \rho)t} \begin{bmatrix} \pi_E\rho \\ 1 \end{bmatrix} + C_2 e^{\pi_E^{-1} rt} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - A^{-1}B =$$
$$= C_1 e^{(\pi_E^{-1} r - \rho)t} \begin{bmatrix} \pi_E\rho \\ 1 \end{bmatrix} + C_2 e^{\pi_E^{-1} rt} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -r^{-1}\kappa \end{bmatrix},$$

as can be checked by heterogeneous ODE system $\dot{x}(t) = \lambda_1 C_1 e^{\lambda_1 t} v_1 + \lambda_2 C_2 e^{\lambda_2 t} v_2 = A[C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 - A^{-1}B] + B = A[C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2]$, whereby $Av_{1,\ 2} = \lambda_{1,\ 2} v_{1,\ 2}$.

As time t converges towards "uncountably positive infinity" so do the scrutinised system's solutions, following the direction dictated by eigenvector v_2; all trajectories moreover diverge from unique[24] equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$. Contrariwise, as time t converges towards "uncountably negative infinity" the scrutinised system's solutions converge towards 0, following the direction dictated by eigenvector v_1.

5.4 Stability analysis. Parameter set Θ was refined to be parameter set $\tilde{\Theta} = \{\pi_E,\ r,\ \rho\} \subset \mathbb{R}_{++}$ such that negative discount rate $\rho \in (0,\ 1)$. One can thus instantiate parametrisation $\pi_E = 1,\ r = 1.02$ and $\rho = 0.99$, whence matrix $A = [(0.03\ 0)\ (-1\ 1.02)]^\top$.

In other words, domestic external inflation π_E is unitary, whereby the growth rate of changed domestic external prices $p_E^{-1}\dot{p}_E = 1 \longrightarrow \dot{p}_E = p_E$, that is, changed domestic external prices $\dot{p}_E$ correspond to

[24]Equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$ is unique also because matrix A, thus parametrised, is invertible.

domestic external prices p_E.

Domestic real interest rate r is similarly indicative of a levy on the principal sum of extant foreign real borrowing $b(t)$ equal to 2%, thereby tracing a hypothetical yearly growth rate in domestic real output y of the same value: $1.02 = 1 + 0.02 = 1 + \frac{2}{100}$, acting as the opportunity cost of investment in foreign real borrowing $\dot{b}(t)$ by relation with the yearly growth rate of domestic real output y.

Domestic negative discount rate ρ lastly equates to 0.99 so that summed instantaneous changes in the domestic country's trade balance utility function valued in the present $\int_0^\infty e^{-\rho t} dU[b(t),\ ex(t)] = \int_0^\infty e^{-\rho t} u[b(t),\ ex(t)]dt$ be marked up at a compound rate almost equal to 100%, since the figurative lending of one's (presently valued, derived) utility to oneself features an opportunity cost approximately correspondent to the entire (presently valued, derived) utility: $\int_0^\infty e^{\rho - \rho t} dU[b(t),\ ex(t)] = \int_0^\infty e^{\rho - \rho t} u[b(t),\ ex(t)]dt = \int_0^\infty dU[b(t),\ ex(t)] = \int_0^\infty u[b(t),\ ex(t)]dt$, being the future value of the summed instantaneous changes in the domestic country's trade balance utility function.

Plugging such values into the general solution of the system in question gives rise to

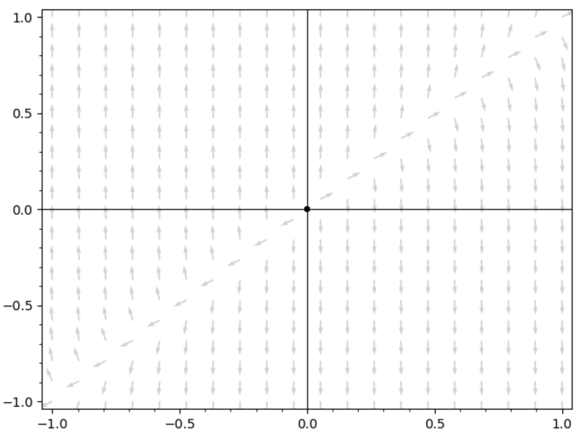
$$x(t) = C_1 e^{0.03t} \begin{bmatrix} 0.99 \\ 1 \end{bmatrix} + C_2 e^{1.02t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.02^{-1}\kappa \end{bmatrix},$$

where constants $C_1,\ C_2,\ \kappa \in \mathbb{R}$. Positive eigenvalues $\lambda_{1,\ 2} = 0.03,\ 1.02$ mean system instability[25] and that unique equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$ is an unstable node.

The direction dictated by eigenvector v_2 is faster than that dictated by eigenvector v_1 on account of the greater eigenvalue λ_2. Correspondingly, the direction dictated by eigenvector v_1 is slower than that dictated by eigenvector v_2 on account of the smaller eigenvalue λ_1.

All trajectories leave the unstable node, being equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$, in the slower direction, save for a single pair of trajectories moving along the faster direction. The (direction field) phase diagram[26] proper to the system in question for equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$ is at last drawn below.

Figure 1: Phase diagram for equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$



Note. This is a (direction field) phase diagram pertaining to the the bi-dimensional, heterogeneous ODE system under scrutiny with parametrisation $\pi_E = 1,\ r = 1.02$ and $\rho = 0.99$ for equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$. In detail, domestic real exports $ex(t)$ are plotted on the horizontal axis, while extant domestic foreign real borrowing $b(t)$ is plotted on the vertical axis. Eigenvector v_1 produces (null cline) equation $b(t) = 0.99^{-1}ex(t)$ and eigenvector v_2 produces (null cline) equation $ex(t) = 0$. Eigenvectors $v_1 = [0.99\ 1]^\top$ and $v_2 = [0\ 1]^\top$ respectively present unstable eigenvalues $\lambda_1 = 0.03$ and $\lambda_2 = 1.02$, determining an outwards flow of all trajectories. Smaller eigenvalue λ_1 is associated with the slower direction of eigenvector v_1 and greater eigenvalue λ_2 is associated with the faster direction of eigenvector v_2. All trajectories leave equilibrium point $[ex(t),\ b(t)] = (0,\ 0)$ by moving along the slower direction, except for a single pair of them in the faster direction, being visibly those immediately parallel to the vertical axis, to the left and the right. Any $[ex(t),\ b(t)]$ point above (null cline) equation $b(t) = 0.99^{-1}ex(t)$ tends towards explosive extant foreign real borrowing $b(t)$, even in the presence of positive domestic real exports $ex(t)$, and any $[ex(t),\ b(t)]$ point below it tends towards implosive extant foreign real borrowing $b(t)$, even in the presence of negative domestic real exports $ex(t)$.

A positive quantity of domestic real exports $ex(t)$ practically speaks to domestic net real exports $nx(t)$ and one of extant foreign real borrowing $b(t)$ practically speaks to extant foreign net real borrowing $enb(t)$. By starting from any point below (null cline) equation[27] $b(t) = 0.99^{-1}ex(t)$ one discerns that, for

[25] https://en.wikipedia.org/wiki/Stability_theory

[26] http://scofield.site/teaching/demos/PhasePortrait2D.html

[27] Null cline equation $b(t) = 0.99^{-1}ex(t)$ stems from $\nabla = \beta = \frac{1-0}{0.99-0} = 0.99^{-1}$ in $b(t) = \alpha + \beta ex(t)$, whence $\alpha = b(t) - \beta ex(t) \longrightarrow \alpha = 1 - 0.99^{-1}0.99 = 1 - 1 = 0 - 0.99^{-1}0 = 0$, so that $b(t) = \alpha + \beta ex(t) = 0.99^{-1}ex(t)$. Null cline equation $ex(t) = 0$ stems from $\nabla = \frac{1-0}{0-0} = \infty$ in $ex(t) = 0$ of necessity.

any quantity of domestic real exports $ex(t)$, even negative, though especially positive, extant foreign real borrowing $b(t)$ is bound to implode, having evermore lent abroad.

Comparably, by starting from any point above (null cline) equation $b(t) = 0.99^{-1}ex(t)$ one discerns that, for any quantity of domestic real exports $ex(t)$, even positive, though especially negative, extant foreign real borrowing $b(t)$ is bound to explode, having evermore borrowed from abroad.

5.5 Initial value problem. Let us impose initial conditions $x(0) = [ex(0)\ b(0)]^\top$ such that first initial condition $ex(0) \leq 0$ and second initial condition $b(0) = 0$. The potential negativity of domestic real exports $ex(t)$ at time period 0 captures the idea whereby the domestic country have already engaged in (net) real importation by the time the analysis begins.

In the face of

$$\begin{aligned} x(t) &= x_c(t) + x_p = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + K = \\ &= C_1 e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + C_2 e^{\lambda_2 t} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} + K = \\ &= C_1 e^{(\pi_E^{-1} r - \rho)t} \begin{bmatrix} \pi_E \rho \\ 1 \end{bmatrix} + C_2 e^{\pi_E^{-1} r t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -r^{-1}\kappa \end{bmatrix} \end{aligned}$$

and parametrisation $\pi_E = 1$, $r = 1.02$ and $\rho = 0.99$ it follows that: (i-a) $ex(0) = C_1 e^{\lambda_1(0)} v_{11} + C_2 e^{\lambda_2(0)} v_{12} + 0 = C_1 v_{11} + C_2 v_{12} = C_1 \pi_E \rho + C_2(0) = C_1 \pi_E \rho < 0 \longrightarrow C_1 < 0$ and (ii-a) $b(0) = C_1 e^{\lambda_1(0)} v_{21} + C_2 e^{\lambda_2(0)} v_{22} - r^{-1}\kappa = C_1 v_{21} + C_2 v_{22} - r^{-1}\kappa = C_1(1) + C_2(1) - r^{-1}\kappa = 0 \longrightarrow C_2 = r^{-1}\kappa - C_1$, where $C_1 < 0$, whence

$$\begin{aligned} ex(t) &= C_1 \pi_E \rho e^{(\pi_E^{-1} r - \rho)t} + (r^{-1}\kappa - C_1)(0) e^{\pi_E^{-1} r t} = C_1 \pi_E \rho e^{(\pi_E^{-1} r - \rho)t} = C_1 0.99 e^{0.03t}, \\ b(t) &= C_1(1) e^{(\pi_E^{-1} r - \rho)t} + (r^{-1}\kappa - C_1)(1) e^{\pi_E^{-1} r t} - r^{-1}\kappa = \\ &= C_1 [e^{(\pi_E^{-1} r - \rho)t} - e^{\pi_E^{-1} r t}] + r^{-1}\kappa(e^{\pi_E^{-1} r t} - 1) = \\ &= C_1 e^{\pi_E^{-1} r t}(e^{-\rho t} - 1) + r^{-1}\kappa(e^{\pi_E^{-1} r t} - 1) = \\ &= C_1 e^{1.02t}(e^{-0.99t} - 1) + (1.02)^{-1}\kappa(e^{1.02t} - 1); \end{aligned}$$

(i-b) $ex(0) = C_1 e^{\lambda_1(0)} v_{11} + C_2 e^{\lambda_2(0)} v_{12} + 0 = C_1 v_{11} + C_2 v_{12} = C_1 \pi_E \rho + C_2(0) = C_1 \pi_E \rho = 0 \longrightarrow C_1 = 0$ and (ii-b) $b(0) = C_1 e^{\lambda_1(0)} v_{21} + C_2 e^{\lambda_2(0)} v_{22} - r^{-1}\kappa = C_1 v_{21} + C_2 v_{22} - r^{-1}\kappa = C_1(1) + C_2(1) - r^{-1}\kappa = 0 \longrightarrow C_2 = r^{-1}\kappa - C_1 = r^{-1}\kappa$, whence

$$\begin{aligned} ex(t) &= (0)\pi_E \rho e^{(\pi_E^{-1} r - \rho)t} + r^{-1}\kappa(0) e^{\pi_E^{-1} r t} = 0, \\ b(t) &= (0)(1) e^{(\pi_E^{-1} r - \rho)t} + r^{-1}\kappa(1) e^{\pi_E^{-1} r t} - r^{-1}\kappa = \\ &= r^{-1}\kappa(e^{\pi_E^{-1} r t} - 1) = (1.02)^{-1}\kappa(e^{1.02t} - 1). \end{aligned}$$

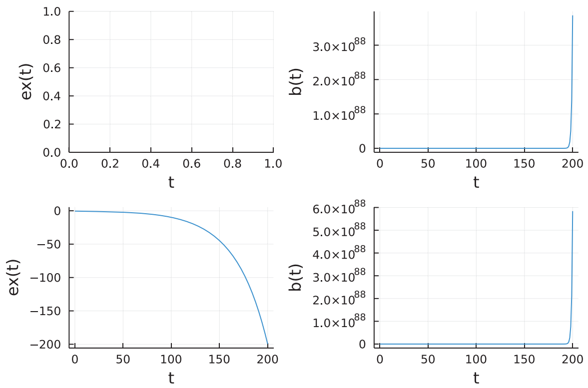
The normalisation of heterogeneous component κ to 1 across both initial value sub-problems and the calibration of solution coefficient C_1 to -0.5 in the first one, as an approximate midpoint between the endpoints of a numerically representative open, real interval between -1 and 0, yields the below plots of the analysed system's solutions at a horizon of 200 quarters (i.e. 50 years): *ceteris paribus,* $\kappa = 1$ for $ex(0) \leq 0$ and $b(0) = 0$; $C_1 = -0.5$, for $ex(0) < 0$ and $b(0) = 0$, since $C_1 \in (-1,\ 0) \subset \mathbb{R}_{--}$ for simplicity.

No domestic real exports $ex(t)$ at time period 0 cause the domestic country not to take up any future real exportation. Negative domestic real exports $ex(t)$ at time period 0, which are domestic real imports $im(t)$, cause the domestic country to take up negative real exportation in the future, which is real importation; the increase in real importation is exponential by definition, gradually accelerating, in detail.

Extant foreign real borrowing $b(t)$ at time period 0 is null for both initial sub-conditions of domestic real exports $ex(t)$; it remains void under both scenarios until slightly before the two hundredth time period, then exploding altogether.

Such a comportment on the part of the analysed solutions strengthens the expected conduct of both domestic real exports $ex(t)$ and extant foreign real borrowing $b(t)$ as laid down by the aforementioned (direction field) phase diagram, by which any $[ex(t),\ b(t)]$ point situated above (null cline) equation $b(t) = 0.99^{-1}ex(t)$, tends towards explosive extant foreign real borrowing $b(t)$, particularly for negative domestic real exports $ex(t)$.

Figure 2: Solution plots of $\dot{x}(t) = A(\Theta)x(t) + B(\Theta)$ with $ex(0) \leq 0$ and $b(0) = 0$



Note. These are the graphs of the solutions to ODE system $\dot{x}(t) = A(\Theta)x(t) + B(\Theta)$ with initial conditions $x(0) = [ex(0)\ b(0)]^\top$ such that first initial condition $ex(0) \leq 0$ and second initial condition $b(0) = 0$, at parametrisation $\pi_E = 1$, $r = 1.02$ and $\rho = 0.99$, in the presence of normalisation $\kappa = 1$, at a 200 quarter horizon. If domestic real exports $ex(t)$ at time period 0 are null, being there none, then the domestic country does not engage in any real exportation at all thenceforth. If domestic real exports $ex(t)$ at time period 0 are negative, being there domestic real imports $im(t)$, then the domestic country engages in negative real exportation thenceforth, being real importation; the decrease in real exportation is clearly exponential, in detail, it is accelerating, but gradual. Extant foreign real borrowing $b(t)$ at time period 0 is always null, being there none until slightly before the two hundredth time period, about which point it explodes altogether, in both instances. Such a pattern on the part of the solutions at hand confirms the expected behaviour of domestic real exports $ex(t)$ and extant foreign real borrowing $b(t)$ whenever situated above (null cline) equation $b(t) = 0.99^{-1}ex(t)$ as point $[ex(t),\ b(t)]$ in the above (direction field) phase diagram, whereby, for negative domestic real exports $ex(t)$, extant foreign real borrowing $b(t)$ eventually explodes.

6. Conclusion

6.1 Russian real trade balance. The upshot of Russia's request to pay for the exportation of its gas in Roubles was an increase in the demand for Roubles on the foreign exchange market and an accompanying real appreciation, which coupled with Russian gas' price inelasticity of demand (i.e. $\eta_{ex_e} = (ex^{-1}e)ex_e < 1$) resulted in an even further increment in the value of exported Russian gas, to the gain of Russia and to the detriment of the EU in particular[28]. Russia also concurrently managed to expand its gas market in China and other emerging countries, mushrooming its real trade balance[29] even more.

6.2 Punchline. The side lesson learnt, in all events, for all its banality, is that the acquisition of the currency demanded by the country from which imports are purchased can only pass through a demand for one's exports by markets in turn, at heart.

This monograph has formalised such and the Triffin dilemma, while innovatively presenting an orderly model of the balance of payments. In such a light it furthermore studied optimal exportation and foreign borrowing through an SDPE model, confirming its global appraisal.

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Appendix

Julia commands for analytical solutions of ODE system $\dot{x}(t) = Ax(t) + B$ with initial conditions $x(0) = [ex(0)\ b(0)]^\top$ such that first initial condition $ex(0) \leq 0$ and second initial condition $b(0) = 0$ (wherein # must replace %).

```
using Plots, SymPy % Suitably add LinearAlgebra, Statistics for more

% DOMAIN, PARAMETERS AND CODOMAINS
t, pi, r, rho, kappa=Sym("t pi r rho kappa"); % Domain and parameters
ex=SymFunction("ex"); % Codomain 1
b=SymFunction("b"); % Codomain 2

% ODE SYSTEM AND SOLUTION
eq1=diff(ex(t), t)-(inv(pi)*r-rho)*ex(t); % Real exports ODE
eq2=diff(b(t), t)+inv(pi)*(ex(t)-r*b(t)-kappa); % Extant foreign real borrowing ODE

sol=dsolve((eq1, eq2)) % ODE system solution

% ODE SYSTEM NUMERICAL SOLUTION
eq1n=eq1(pi=>1, r=>1.02, rho=>0.99); % Real exports ODE (parametrised)
eq2n=eq2(pi=>1, r=>1.02, kappa=>1); % Extant foreign real borrowing ODE (fully parametrised)
soln=dsolve((eq1n, eq2n)) % ODE system solution (fully parametrised)

% ODE SYSTEM NUMERICAL SOLUTION (ALTERNATIVE)
C1, C2=Sym("C1 C2"); % ODE system solution coefficients
v1=[0.99; 1]; % ODE system solution eigenvector 1 taken from soln
v2=[0; 1]; % ODE system solution eigenvector 2 taken from soln
A=[0.03 0; -1 1.02]; % ODE system companion matrix (parametrised)
B=[0; kappa]; % ODE system heterogenous component matrix
```

```
x=C1*exp(0.03*t)*v1+C2*exp(1.02*t)*v2-inv(A)*B; % ODE system solution (parametrised)

Bn=[0; 1]; % ODE system heterogenous component matrix, with kappa=1
xn=C1*exp(0.03*t)*v1+C2*exp(1.02*t)*v2-inv(A)*Bn % ODE system solution (fully parametrised)

% INITIAL VALUE PROBLEM
ex_0=xn[1, 1](t=>0); % ex(0)
C1_0=first(solve(ex_0, C1)); % C1 derived from initial sub-condition ex(0)=0
b_01=xn[2, 1](t=>0, C1=>0); % b(0), with C1 taken from C1_0
C2_01=first(solve(b_01, C2)); % C2 derived from initial condition b(0)=0, with C1 taken ...
    from C1_0
xi1=C1_0*exp(0.03*t)*v1+C2_01*exp(1.02*t)*v2-inv(A)*Bn % ODE system solution (fully ...
    parametrised, with initial conditions)

% INITIAL VALUE PROBLEM (ALTERNATIVE)
b_02=xn[2, 1](t=>0, C1=>-0.5); % b(0), with C1=-0.5 imposed from initial sub-condition ...
    ex(0)<0
C2_02=first(solve(b_02, C2)); % C2 derived from initial condition b(0)=0, with C1=-0.5 ...
    imposed from initial sub-condition ex(0)<0
xi2=-0.5*exp(0.03*t)*v1+C2_02*exp(1.02*t)*v2-inv(A)*Bn % ODE system solution (fully ...
    parametrised, with alternative initial conditions)

% SOLUTION GRAPHS
t=0:200; % Time period domain (200 quarters, 50 years)
p1=plot(0, t, ylabel="ex(t)", xlabel="t", label=""); % where xi1[1, 1]=0
p2=plot(xi1[2, 1], t, ylabel="b(t)", xlabel="t", label="");
p3=plot(xi2[1, 1], t, ylabel="ex(t)", xlabel="t", label="");
p4=plot(xi2[2, 1], t, ylabel="b(t)", xlabel="t", label="");
plot(p1, p2, p3, p4, layout=(2, 2), label="");
savefig("fiJ.pdf")
```