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Abstract

We propose a multi-stage stochastic trading cost model in optimal portfolio selection. This strategy captures uncertainty in implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. We assume that implicit costs are stochastic as are asset returns. We use mean absolute deviation as our risk and apply the model to securities on the Johannesburg Stock Market. The model generates optimal portfolios by minimizing total implicit transaction costs incurred. It provides least-cost optimal portfolios whose net wealths are better than those generated by the mean-variance, minimax and mean absolute deviation models.

Key words: implicit transaction costs, stochastic programming.

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1 Introduction

Financial markets are inherently volatile, characterized by shifting values, risks and opportunities. The prices of individual securities are frequently changing for numerous reasons that include shifts in perceived value, localized supply and demand imbalances, and price changes in other sector investments or the market as a whole. Reduced liquidity results in price volatility and market risk to any contemplated transaction. As a result of this volatility, transaction cost analysis (TCA) has become increasingly important in helping firms measure how effectively both perceived and actual portfolio orders are executed. The increasing complexities and inherent uncertainties in financial markets have led to the need for mathematical models supporting decision-making processes. In this study, we propose a stochastic multi-stage mean absolute deviation model with trading costs (SMADTC) that minimizes implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. This is achieved by allowing the investor to choose his or her desired implicit transaction cost value and portfolio mean rate of return or risk level, where the risk is defined by the mean absolute deviation of assets' returns from expected portfolio return. The multi-stage stochastic transaction cost model captures assets' returns, implicit transaction costs and risk due to uncertainty. We apply stochastic programming since it has a number of advantages over other techniques. Firstly, stochastic programming models can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for securities' returns, but we can rely on the empirical distribution of these returns. Secondly, they can address practical issues such as transaction costs, turnover constraints, limits on securities and prohibition of short-selling. Regulatory and institutional or market-specific constraints can be accommodated. Thirdly, they can flexibly use different risk measures.

Konno and Yamazaki [14] propose the mean absolute deviation (MAD) model, in deterministic form, as an alternative to the mean-variance (MV) model by Markowitz [17]. MAD is a dispersion-type risk linear programming (LP) computable measure that may be taken as an approximation of the variance when the absolute values replace the squares. It is equivalent to the mean-variance if the assets' returns are multivariate normally distributed. However, using a linear model considerably reduces the time needed to reach a solution, thus making the MAD model more appropriate for large-scale portfolio selection. It makes intensive calculations of the covariance matrix unnecessary as opposed to the mean-variance model. The MAD model is also sensitive to outliers in historical data (Byrne and Lee, [2]) Much financial research has been done regarding asset allocation, portfolio construction and performance attribution. However, for the total performance of a portfolio, the quality of the implementation is as important as the decision itself. Implementation costs usually reduce portfolio returns with limited potential to generate upside potential. A portfolio must strike what an investor believes to be an acceptable balance between risk and reward, having considered all costs incurred in the setting or rebalancing of the portfolio. Investment portfolios should be rebalanced to take account of changing market conditions and changes in funding. This brings with it some trading costs, which can be either direct or indirect. Direct trading costs are observable and they include brokerage commissions, market fees and taxes. Indirect costs are invisible and these include bid-ask spread, market impact and opportunity costs.

Of the literature that is devoted to modeling portfolio selection with transaction costs, the greater part concentrates on proportional transaction costs. Kozmik [15] discusses an asset allocation strategy with transaction costs formulated as a multi-stage stochastic programming model. He considers transaction costs as proportional to the value of the assets bought or sold, but does not consider implicit transaction costs in the model. He employs Conditional-Value-at-Risk as a risk measure. Moallemi and Saglam [18] study dynamic portfolio selection models with Gausian uncertainty using linear decision models incorporating proportional transaction costs. They assume that trading costs such as bid-ask spread, broker commissions and exchange fees are proportional to the trade size. However, as assets' prices follow a random-walk process, such price movement would result in randomly fluctuating transaction costs due to a number of factors that include asset liquidity, market impact and so on. In economic recessions and booms where asset returns are characterised by extreme movements, the extreme movements of the market are not always reflected in all individual stocks. Some individual stocks show an extreme reaction while others exhibit a milder reaction (Jansen and De Vries, [12]). Hence considering proportional transaction costs in an uncertain environment does not provide good estimate of trading costs, especially implicit transaction costs. Lynch and Tan [16] study portfolio selection problems with multiple risky assets. They develop analytic frameworks for the case with many assets taking proportional transaction costs. Xiao and Tian [20] estimate implicit transaction costs in Shenzhen A-stock market using the daily closing prices, and examine the variation of the cost of Shenzhen A-stock market from 1992 to 2010. They use the Bayesian Gibbs sampling method proposed by Hasbrouck [10] to analyze implicit costs in the bull and bear markets. Hasbrouck [10] incorporates the Gibbs estimates into asset pricing specifications over a historical sample and find that effective cost is positively related to stock returns. Brown and Smith [3] study the problem of dynamic portfolio optimization in a discrete-time finite-horizon setting, and they also take into account proportional transaction costs. Cai, et al [4] examine numerical solutions of dynamic portfolio optimization with transaction costs. They consider proportional transaction costs which can be either explicit or implicit, whichever is greater. However, transaction cost analysis requires the identification of the type of cost to be estimated in order to explore effective ways of having a good estimate of it, hence enabling an investor to make an informed decision. Thus, in our study, we concentrate on implicit transaction costs as these are invisible and can easily erode the profits of an investment. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (Hondt and Giraud, [5]). Konno and Wajayanayake [13] propose the deterministic mean absolute deviation model with transaction costs modeled by a concave function. They use a linear cost function as an approximation to the concave cost function. Gulpinar, et al [6] propose a multi-stage mean-variance portfolio analysis with proportional transaction costs. Our study considers random transaction costs since asset prices are random.

Some investors do not like too high costs as these are known to erode the profits of investment. Most models in the literature are static models and they are essentially single-period models. There is only one decision to be made, for the first period. In real-life, decisions are made with possibilities of adjustment down the road, since the future is unknown. Stochastic programming models hence are dynamic, covering multiple-time periods with associated separate decisions, and they account for the stochastic decision process. The main features of stochastic programming are scenarios and stages. The uncertainty about future events is captured by a set of scenarios, which is a representative and comprehensive set of possible realizations of the future. Stochastic programming recognizes that future decisions happen in stages, incorporating earlier decisions and events that occurred during earlier time-periods.

The main contributions of this study include:

(i) The development of a stochastic multi-stage trading cost model that generates optimal portfolios while minimizing uncertain implicit transaction costs incurred by an investor during initial trading and in subsequent rebalancing of portfolios;

(ii) The development of a strategy that captures uncertainty in stock prices and in corresponding implicit trading costs by way of scenarios.

This paper is organized as follows. In Section 2, we discuss the formulation of the stochastic trading cost model for multi-stage optimal portfolio selection. Transaction costs and portfolio rebalancing constraints are explained. Discussion on transaction cost calculation is given in this section. In Section 3, we demonstrate the application of the model to securities taken from the Johannesburg Stock Market courtesy of I-Net Bridge. We compare the model's performance with the performance of the mean-variance, minimax (MM) and mean absolute deviation models. We conclude, in Section 4, by giving a summary of our findings and the main contributions of this study, as well as directions on future research.

2 Materials and methods

We determine a multi-period discrete-time optimal portfolio strategy over a given investment horizon T. The period T is divided into two discrete intervals T_1 and T_2. T_1 defines the planning phase, where $T_1 = 0, 1, \cdots, \tau$. Period $T_2 = \tau + 1, \cdots, T$ is the period to investment maturity. During T_1, an investor makes decisions and adjustments to his portfolio at each of the τ periods as some assets' returns get realized. The initial investment takes place at $t = 0$, with portfolio restructuring at times $t = 1, 2, \cdots, \tau$. After period τ, no further decisions are implemented until investment maturity at $t = T$. A portfolio is restructured in terms of asset return and risk which is measured by the mean absolute deviation of assets' returns from expected portfolio return. This restructuring of the portfolio brings with it transaction costs, as the investor buys or sells shares of some securities to balance his portfolio. For a conservative investor, trade execution costs are paramount. Hence the need to minimize these costs. We thus consider an investor who is interested in getting an optimal portfolio while at the same time keeping transaction costs to the minimum.

2.1 Scenario Generation

Let $\mathbf{R} = \{R_1, \cdots, R_t\}$ be stochastic events at $t = 1, \cdots, T$, and consider $I = \{i : i = 1, 2, \cdots, n\}$ to be a set of securities for an investment. The decision process is non-anticipative (i.e., a decision at a given stage does not depend on the future realization of the random events). The decision at period t is dependent on R_{t-1}. We define a scenario as a possible realization of the stochastic variables $\{R_1, R_2, \cdots, R_t\}$. Hence the set of scenarios corresponds to the set of paths followed from the root to the leaves of a tree, S_τ, and nodes of the tree at level $t \geq 1$ corresponds to possible realizations of R_t. Each node at a level t corresponds to a decision which must be determined at time t, and depends in general on R_t, the initial wealth of the portfolio and past decisions. Given the event history up to time t, R_t, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observations R_{t+1}. The branching process is represented by a scenario tree. An example of a scenario tree with 2 time periods and three-three branching structure is shown in Figure 1 below.

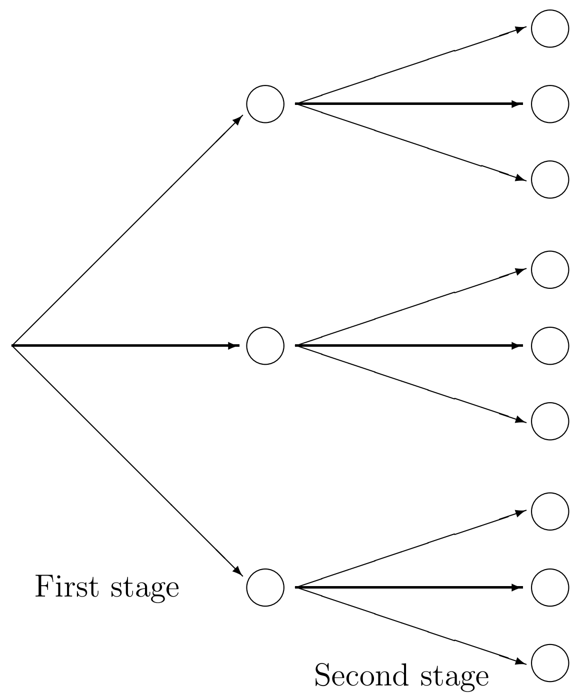

Figure 1: Scenario tree

The uncertain return of the portfolio at the end of the period t is $R = R(x_t, r_t)$. This is a random variable with a distribution function, say F, given by

$$F(x, \mu) = p\{R(x, r) \leq \mu\}$$

The expected return of the portfolio at the end of period t is

$$r_{pt} = E[R(x_t, R_t)] = r(x_t, R_t).$$

Suppose the uncertain returns of the assets, R_t, in period t are represented by a finite set of discrete scenarios $\Omega = \{s : s = 1, 2, \cdots, S\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $R_s = (R_{1s}, R_{2s}, \cdots, R_{ns})^T$ with associated probability $p_s > 0$, where $\sum_{s\in\Omega} p_s = 1$. The portfolio return under a particular realization of asset return R_s of period t is denoted by $r_{st} = r(x_t, R_{st})$. The expected portfolio return of period t is now given by

$$r_{pt} = E[r(x_t, R_{st})] = \textstyle\sum_{s\in\Omega} p_s r(x_t, R_{st})$$

2.2 Model constraints

The dynamic portfolio selection in discrete time allows an investor to adjust dynamically his or her portfolio at successive stages. We assume that the investor joins the market at $t = 0$ with initial wealth W_0, and W_t being portfolio wealth at period t. The wealth can be partitioned among the n-assets at the beginning of each of the τ consecutive time periods to rebalance the portfolio. The investor is seeking an optimal investment strategy,

$x_t = [x_{1st}, x_{2st}, \cdots, x_{nst}]$ for $t = 1, 2, \cdots, \tau$, such that

$$\textstyle\sum_{i=1}^{n} x_{it} = \sum_{s=1}^{S} x_{ist} = 1, t = 1, 2, \cdots, \tau.$$

where S is the total number of scenarios in period t. Let a_{ist} and v_{ist} be, respectively, the buying and selling proportions of asset i of scenario s of period t. Then we have $a_{ist} = \frac{A_{ist}}{W_t}$ and $v_{ist} = \frac{V_{ist}}{W_t}$ where A_{ist} is the amount of money used to buy new shares of asset i of scenario s in period t, and V_{ist} is the money obtained from selling shares of asset i of scenario s in period t. Thus, in portfolio rebalancing at the beginning of each time period t, we have

$$x_{i,s,t} = x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \quad (1)$$

We remark here that either a_{ist} or v_{ist} is zero at each rebalancing of the portfolio since we cannot buy and sell an asset at the same time. This results in the constraint

$$a_{ist} \cdot v_{ist} = 0 \quad (2)$$

We assume that the portfolio is self-financing and consider that money can only be added at $t = 0$ and not in subsequent periods.

The decision made by the investor at period t depends on $x_{i,s,t-1}$ and the yield of the investment of asset i of scenario s. Thus, the expected return of asset i in period t is given by

$$\begin{aligned} r_{it} &= \sum_{s \in Q} p_s \cdot R_{i,s,t} \cdot (x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}), i = 1, \cdots, n; t = 1, \cdots, \tau. \\ &= \sum_{s \in Q} p_s \cdot R_{ist} \cdot x_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau. \quad (3) \end{aligned}$$

for all scenarios s of asset i. Here, $Q \subset \Omega$ is a set of scenarios of asset i of period t. When rebalancing the portfolio, it is observed that

$$0 \leq \sum_{i} a_{ist} \leq \sum_{j,j\neq i} v_{ist}; i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (4)$$

This constraint ensures that the amount of money used to purchase shares of assets i should be at most equal to the amount of money obtained from selling shares of assets $j, (j \neq i)$ of period t during portfolio rebalancing.

The following constraint guarantees that the proportion of asset i of scenario s in period t sold for portfolio rebalancing should be at most the proportion of the asset in the portfolio,

$$0 \leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (5)$$

The inequality (5) also ensures that no short-selling takes place. To provide for diversification of the portfolio, bounds U_{ist} on decision variables are considered resulting in the constraint

$$0 \leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau. \qquad (6)$$

It should be noted that scenarios may reveal identical value for the uncertain quantities up to a certain period. These scenarios that share common information must yield the same decisions up to that period. Thus we have the constraint

$$x_{ist} = x_{is't} \tag{7}$$

for all scenarios s and s' with identical past up to time t.

2.3 Expected portfolio Wealth

The investor aims to obtain an optimal strategy that minimizes transaction costs. Let k_{ist} and l_{ist} be, respectively, the rate of buying and selling transaction costs of the quantity of asset i of scenario s bought or sold for portfolio rebalancing at the beginning of period t. Thus, the transaction cost rate incurred by the investor for buying or selling proportions a_{ist} or v_{ist} respectively, of asset i of scenario s in period t is given by

$$k_{ist}a_{ist} + l_{ist}v_{ist}$$

We remark here that since $a_{ist} \cdot v_{ist} = 0$ (from (2)), we have either

$$k_{ist}a_{ist} = 0 \quad \text{or} \quad l_{ist}v_{ist} = 0 \tag{8}$$

or both being equal to zero. Thus, the expected transaction cost rate of the portfolio in period t becomes

$$\textstyle\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\}, i = 1, 2, \cdots, n; t = 1, 2, \cdots, \tau.$$

The gross mean rate of return of the portfolio of period t is given by

$$\begin{aligned} r_{pt} &= \sum_{s=1}^{S} p_s \cdot R_{i,s,t} \cdot (x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}), i = 1, \cdots, n; t = 1, \cdots, \tau. \\ &= \sum_{s=1}^{S} p_s \cdot R_{ist} \cdot x_{ist}, i = 1, \cdots, n; t = 1, \cdots, \tau. \end{aligned} \tag{9}$$

since we have a total of S scenarios in each period. The wealth of period t, without transaction costs, becomes

$$W_t = (1 + r_{pt}) \cdot W_{t-1}, t = 1, \cdots, \tau \tag{10}$$

Denoting the net expected portfolio return rate of period t by N_{pt}, we get

$$N_{pt} = r_{pt} - \textstyle\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}.$$

Thus, the portfolio wealth of period t taking transaction costs into account is given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, t = 1, \cdots, \tau. \tag{11}$$

2.4 Expected portfolio Risk

The portfolio risk for any realization of any period is measured by the mean absolute deviation of the realized returns relative to the expected portfolio return, r_{pt}. We consider deviation of a scenario return from expected portfolio return at any period t. Konno and Yamazaki [14] develop the deterministic mean absolute deviation model in an attempt to improve on the famous Markowitz [17] mean-variance model. Their mean absolute deviation model has portfolio risk expressed as

$$H(p) = \frac{1}{\mathrm{T}} \sum_{t=1}^{T} \left| \sum_{i=1}^{n} (r_{it} - r_i) x_i \right|$$

where r_{it} is the realized return of asset i of period t, r_i is the expected return of asset i per period, and x_i is the proportion of wealth invested in asset i. We therefore extend this formulation of portfolio risk by taking into account uncertainty of asset returns and transaction costs in portfolio rebalancing. Using the deterministic model as our basis, three key stochastic framework elements are incorporated to formulate the stochastic mean absolute deviation model. The first element considered is the concept of scenarios. Since the deterministic model represents one particular scenario, the inclusion of multiple scenarios in capturing uncertainty result in the increase in the number of variable parameters. Thus uncertainty is represented by a set of distinct realizations $s \in \Omega$. We now consider the scenario parameter, along with other parameters of security and time period. Secondly, the stochastic trading cost model allows for the implementation of recourse decisions as unfolding information on assets' returns get realized. The third element is the probabilistic feature of the stochastic framework which assigns probabilities to scenarios. The parameter p_s represents scenario probability. Scenarios may reveal identical value for the uncertain quantities up to a certain period. Scenarios that share common information must yield the same decisions up to that period. The stochastic mean absolute deviation also incorporates transaction costs incurred during portfolio rebalancing at each time period. Thus, we obtain the following portfolio risk:

$$H(r_{pt}) = \frac{1}{\tau} \sum_{t=1}^{\tau} | \sum_{s=1}^{S} p_s (R_{st} - r_{pt}) x_{st} | \tag{12}$$

where R_{st}, r_{pt} and x_{st} are as defined earlier. Let us denote $y_{st} = (R_{ist} - r_{pt})x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau$. Since $R_{i,s,t} = R_{i,s,t}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$ and $r_{pt} = r_{pt}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$, we have $y_{s,t} = y_{s,t}(x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t})$ as well. Equivalently, we have $y_{st} = y_{st}(x_{ist})$. Thus the portfolio risk becomes

$$H(r_{pt}) = \frac{1}{\tau} \sum_{t=1}^{\tau} | \sum_{s=1}^{S} p_s y_{st} |$$

This risk function is non-linear, and since we intend to have a linear programming model we linearize the portfolio risk function and it becomes

$$H(r_{pt}) = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t \tag{13}$$

where $Z_t \geq | \sum_{s=1}^{S} p_s y_{st} |$.

2.5 Multi-stage Stochastic Transaction Cost Model

We present the multi-stage stochastic transaction cost model in optimal portfolio selection as a minimization of implicit trading costs incurred by an investor during initial trading and in subsequent rebalancing of the portfolio. We subject the model to constraints describing the growth of the portfolio in all scenarios, some performance constraints and bounds on variables.

We constrain the final expected wealth to be at least a particular value, say α, desired by the investor. The portfolio risk, measured by the stochastic mean absolute deviation, is also constrained to be at most a value chosen by the investor, say θ_t, at any period t. The optimization model provides an optimal investment strategy that minimizes implicit transaction costs while achieving the specified expected wealth and desired risk level. Varying the expected return or risk and re-optimizing generate a set of optimal portfolios, forming the efficient frontier. If we let the investor's desired minimum net portfolio return to be at least λ say, we state the stochastic multi-stage transaction cost model as follows:

Minimize

$$\sum_{s=1}^{S} p_s\{k_{ist}a_{ist} + l_{ist}v_{ist}\} \tag{14}$$

subject to

$$\begin{aligned}
0 &\leq Z_t + \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq Z_t - \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
N_{pt} &\geq \lambda_t \\
W_t &= (1 + N_{pt})W_{t-1}, t = 1, \cdots, \tau, \\
\theta_t &\geq \frac{1}{\tau}\sum_{t=1}^{\tau} Z_t \\
1 &= \sum_{s=1}^{S} x_{ist}, i = 1 \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq \sum_{i} a_{ist} \leq \sum_{j=1, j\neq i}^{n} v_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &= a_{ist} \cdot v_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
x_{ist} &= x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}$$

The first two constraints ensure that the deviation is absolute. In addition to constraining the final rate of return of the portfolio, constraints of the form

$$N_{pt} \geq \lambda_t, t = 1, \cdots, \tau,$$

can be added to ensure any desired intermediate expected performance. We assume that no borrowing is done and the portfolio is self-financing. It deserves mention that the existence of a riskless asset among the securities is regarded as a special case in the stochastic transaction cost model formulation (14) above.

It has been noted earlier that for each asset i, a_{ist} and v_{ist} cannot simultaneously be non-zero. Thus, we can now state problem (14) without the complementary constraint $a_{ist} \cdot v_{ist} = 0$ as the following linear program.

Minimize

$$C_{pt} = \sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\} \tag{15}$$

subject to

$$
\begin{aligned}
0 &\leq Z_t + \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq Z_t - \sum_{s=1}^{S} p_s y_{st}, i = 1, \cdots, n; t = 1, \cdots, \tau, \\
N_{pt} &\geq \lambda_t \\
W_t &= (1 + N_{pt}) W_{t-1}, t = 1, \cdots, \tau, \\
\theta_t &\geq \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t \\
1 &= \sum_{s=1}^{S} x_{ist}, i = 1 \cdots, n; t = 1, \cdots, \tau, \\
0 &\leq \sum_{i} a_{ist} \leq \sum_{j=1, j \neq i}^{n} v_{ist}, s = 1, \cdots, S; t = 1, \cdots, \tau \\
0 &\leq v_{ist} \leq x_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
0 &\leq x_{ist} \leq U_{ist}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau, \\
x_{ist} &= x_{is't}, i = 1, \cdots, n; s = 1, \cdots, S; t = 1, \cdots, \tau,
\end{aligned}
$$

where C_{pt} is the total portfolio transaction cost rate at time t. Alternatively, the objective function can be expressed as minimizing the total portfolio transaction cost in period t as

$$C = [\textstyle\sum_{s=1}^{S} p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}] \cdot W_{t-1}$$

2.6 Transaction cost measurement

An investor incurs transaction costs when buying or selling securities on the Stock Market. This takes place during initial trading and when rebalancing the portfolio in subsequent periods. Transaction costs are either explicit or implicit. Explicit costs are directly observable, and they include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. These costs do not rely on the trading strategy and can easily be determined before the execution of the trade. However, implicit costs are embedded in the stock price, and hence are invisible. They depend mainly on the trade characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of trade execution.These can broadly be put into three categories, namely market impact, opportunity costs and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (Hondt and Giraud, [5]). When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed' price that would have prevailed if the trade had not occurred.

Thus we assume immediate trade execution and regard market impact to account for the total implicit costs. We follow implicit transaction cost calculation as provided by Hau [9]. We use the spread mid-point benchmark and take the transaction price to be the last price of the month. We calculate the effective spread as twice the distance from the mid-price measured in basis points. Thus, for the mid-price P^M, mid-point of the bid-ask spread, and transaction price P^T, we obtain the effective spread (implicit transaction cost) as

$$SPREAD^{Trade} = 200 \times \frac{|P^T - P^M|}{P^M}$$

We consider the transaction price to be the last price of the month for each security. These are prices of securities traded on the Johannesburg Stock Market from 1 January 2008 to 30 September 2012. We got the historical data courtesy of I-Net Bridge.

3 Results and discussion

We use monthly historical data of securities on the Johannesburg Stock Market from January 2008 to September 2012. We evaluate the performance of the proposed SMADTC model by comparing portfolios developed from it and the mean-variance, mean absolute deviation and minimax (MM) models. These models are shown in appendix section. The following criteria are used in the selection of stocks to comprise our initial portfolio:

(a) Stocks with negative mean returns for the entire period considered are excluded from the sample,

(b) Companies which were not on the list by January 2008 and only entered the JSE afterwards are excluded.

(c) Those assets having the highest positive mean returns are taken to become our initial portfolio.

We take empirical distributions computed from past monthly returns as equiprobable scenarios. A scenario, R_{ist}, for the return of asset i of period t is obtained as

$$R_{ist} = \frac{P_{i,s,t}}{P_{i,s,t-1}} - 1$$

where $P_{i,s,t}$ is a historical monthly price of asset i.

In the first stage of analysis, 13 securities are chosen to comprise our initial portfolio. The mean returns, total transaction costs and total wealth of each of the four optimal portfolios developed according to models15, A_1, A_2 and A_3 are calculated and then compared.

In using the proposed model, we consider five scenarios for each asset return and the corresponding transaction cost, and apply the SMADTC model over two stages. However, comparison with other models is restricted to the first stage. We take empirical distributions of the 13 securities comprising our initial portfolio. Since for each security we have 54 monthly returns, we number the months from 1 to 54, and use random numbers to select asset returns and associated transaction costs corresponding to scenarios of a security. We take implicit transaction costs calculated from the effective bid-ask spread corresponding to each selected asset return. We assume that these transaction costs are random since they are randomly selected together with corresponding asset' returns. Thus, a scenario comprises an asset return and the associated transaction cost. The implicit cost for each of the 13 assets is taken into account as it is in the buying or selling of each asset that the cost is incurred. We give the transaction cost as a rate. We take each scenario to be equally likely to occur.

It should be noted that scenarios are only considered for the proposed SMADTC model. Other models use mean returns and risks calculated over the whole period under examination. First-stage optimal portfolios of all models are compared. For the SMADTC model, we consider five scenarios for each asset return in our demonstration, each with a probability of occurring of $\frac{1}{5}$, and apply the model over two stages. This consideration is taken noting that in stochastic programming the scenario tree grows exponentially. At the end of the first stage, an investor decides on his first-stage optimal portfolio as given by the investor's chosen portfolio transaction cost, the diversification limit, the gross portfolio mean return or the net portfolio mean return as the case may be, and the associated risk given by the mean absolute deviation.

3.1 Analysis of results

Results of the SMADTC model are given in stages.

Stage 1

We consider an investor who has R10000 to spend on his initial portfolio. The optimal portfolios generated by the four models are shown in Table 1 below. The phrase 'D.L' stands for 'diversification limit'.

The information in Table 1 reveals that the portfolios generated by the MM model have the highest gross mean returns for each diversification limit considered, followed by the MV model. The SMADTC model generates optimal portfolios with least gross mean returns, up to when the diversification limit is 0.25. The portfolio gross mean returns from the MAD and the MV models show a decreasing trend as the portfolios become less diversified. On the contrary, the MM and SMADTC generated portfolios have gross mean returns that increase as portfolios become less diversified.

Although the MM model appears to generate optimal portfolios with highest gross wealth, it is evident that these portfolios also have the highest implicit transaction costs, resulting in the least net portfolio wealth. The SMADTC model generates optimal portfolios with the least transaction costs per each diversification limit. The MAD model portfolios have transaction costs that start being greater than those of MV portfolios, but becoming less as portfolios become less diversified.

The greatest advantage of the SMADTC lies in its ability to generate an optimal portfolio whose net wealth is always more than amount invested. The other models generate portfolios which are not optimal investments as intended because of the eroding effect of implicit transaction costs. The information also reveals that implicit transaction costs in SMADTC portfolios decline as the portfolio becomes less diversified.

Table 1. Summary Statistics of Optimal Portfolios

D.L	Model	Mean return	Risk	Gross Wealth	Cost	Net Wealth
0.10	MAD	0.027	0.002	10269.80	432.00	9837.80
	MV	0.029	0.002	10289.02	426.34	9862.68
	MM	0.031	0.031	10314.80	502.10	9812.70
	SMADTC	0.014	0.001	10138.09	3.71	10134.38
0.15	MAD	0.025	0.022	10251.75	479.10	9772.65
	MV	0.028	0.002	10277.97	397.32	9880.65
	MM	0.033	0.033	10332.30	411.15	9921.15
	SMADTC	0.018	0.005	10174.99	2.81	10172.18
0.20	MAD	0.024	0.023	10240.20	223.20	10017.00
	MV	0.027	0.002	10268.25	378.64	9889.61
	MM	0.035	0.035	10346.80	503.60	9843.20
	SMADTC	0.019	0.006	10187.44	2.16	10185.28
0.25	MAD	0.023	0.025	10229.25	248.50	9980.75
	MV	0.026	0.002	10259.68	356.71	9902.97
	MM	0.036	0.036	10356.25	591.00	9765.25
	SMADTC	0.020	0.007	10194.80	1.70	10193.10

0.30	MAD	0.023	0.025	10221.70	272.80	9948.90
	MV	0.026	0.002	10255.02	340.06	9914.96
	MM	0.036	0.036	10362.50	550.00	9812.50
	SMADTC	0.023	0.010	10226.24	1.56	10224.68
0.35	MAD	0.021	0.026	10214.90	270.55	9944.35
	MV	0.025	0.002	10250.44	318.87	9931.57
	MM	0.037	0.037	10368.30	502.10	9866.20
	SMADTC	0.025	0.012	10249.58	1.42	10248.16
0.40	MAD	0.021	0.027	10209.60	215.20	9994.40
	MV	0.025	0.002	10247.81	306.70	9941.12
	MM	0.037	0.037	10373.20	440.40	9932.80
	SMADTC	0.026	0.013	10256.72	1.28	10255.44

However, portfolio risk increases instead. It is also evident that portfolio wealth increases with decrease in portfolio diversification.The wealth from a portfolio generated by SMADTC or MV model grows in value as the portfolio become less diversified. This is not the case with MAD and MM optimally generated portfolios whose wealths decline with decrease in diversification.

The assets' weights of portfolios generated by the four models are compared for each diversification level. Assets considered are numbered $A_1, \cdots, A_{13}$ and their descriptions are given in Appendix section. This analysis is shown in Table 2. It is evident that as the portfolios become less diversified, the assets in optimal portfolios become different, with the exception of MAD and MV generated portfolios that seem to maintain the same assets. While the SMADTC, MAD and MM models diversify portfolios as per the diversification level, there is little influence of such a restriction on assets in optimal portfolios generated by MV model. It is noted that the MV optimal portfolios comprise more assets in relatively smaller quantities than any portfolio generated by other models.

Table 2.Assets' Percentage Compositions in Optimal Portfolios.

D.L	Model	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}
0.10	MAD	10	10	10	10	10			10	10	10		10	10
	MV	10	7.8	10	10	10	4.5	3.9	3.7	10	9.2	10	10	10
	MM	10	10	10		10	10	10	10		10	10		10
	SMADTC	10	10	10	10		10	10		10	10		10	10
0.15	MAD	15	15	15	15					15	10		15	
	MV	15	6.1	13.7	15	6.4			2.9	15	7.3	14.7	0.6	3.2
	MM					15	15	15	15		10	15		15
	SMADTC		15	15				15		15	15		10	15
0.20	MAD	20		20	20					20			20	
	MV	20	4.0	10.8	20	3.1			3.4	20	4.4	13.1		1.3
	MM						20	20	20			20		20
	SMADTC		20	20				20		20	20			
0.25	MAD	25			25					25			25	
	MV	24.5	2.3	8.7	22.3	0.3			3.9	25	1.1	11.8		
	MM						25	25	25			25		
	SMADTC		25	25						25	25			

0.30	MAD	10			30				30			30
	MV	22.8	1.8	8.6	21			3.9	30		11.9	
	MM					30	30	10			30	
	SMADTC		30	30					10	30		
0.35	MAD				30				35			35
	MV	20.5	1.2	8.1	19.5			3.8	35		11.8	
	MM					30	35				35	
	SMADTC		35	30						35		

Stage 2

The MV, MM and MAD models are not considered in this stage. It is only the SMADTC model that is examined to assess its multi-stage potential in generating competent optimal portfolios. We consider a first-stage optimal portfolio with six securities. The second-stage optimal portfolios generated by the model reveal that as the investor diversifies his or her portfolio, implicit transaction costs increase while the portfolio wealth decreases, as shown in Table 3 below. This can be attributed to an increasing number of securities being included in the portfolio as the diversification limit decreases.

Table 3. Second-stage Optimal Portfolios

D.L	Gross mean	Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
0.200	0.006	5	0.005	0.004	4.222	10240.097
0.225	0.007	5	0.006	0.005	3.683	10248.122
0.250	0.007	4	0.007	0.006	3.143	10256.147
0.275	0.008	4	0.008	0.006	2.936	10263.840
0.300	0.009	4	0.009	0.007	2.728	10271.534
0.325	0.010	4	0.009	0.008	2.520	10279.227
0.350	0.010	3	0.010	0.009	2.318	10286.915

A further analysis is done by holding the diversification limit constant and varying the portfolio gross mean return as shown in Table 4. Here, 'assets' imply 'number of assets'. The results show that for the same portfolio gross mean return that can be achieved at different diversification limits, implicit transaction costs decrease with increasing diversification limit. The portfolio wealth also increases with increasing diversification limit. For the same risk, we observe that the greater the diversification limit the lower the portfolio implicit transaction cost. It is also noted that for each diversification limit considered, an increase in portfolio gross mean return causes implicit transaction costs to rise.

Table 4. Stage (2) Optimal Portfolios: Portfolio rate of return constrained

D.L	Gross mean	No. of Assets	Net mean	Risk(MAD)	Cost	Wealth
0.175	0.005	6	0.005	0.003	5.402	10231.432
	0.006	6	0.005	0.004	5.479	10240.062
	0.007	6	0.006	0.005	5.623	10250.103
	0.008	6	0.007	0.006	5.814	10260.096
	0.009	6	0.008	0.007	6.010	10270.085
	0.010	6	0.009	0.008	6.210	10290.069
	0.011	6	0.010	0.009	6.611	10289.853
	0.012	6	0.011	0.010	7.308	10299.341
	0.013	6	0.012	0.011	8.036	10308.797
	0.014	6	0.013	0.012	8.837	10318.180
	0.015	6	0.014	0.013	9.644	10327.557
	0.016	6	0.015	0.014	11.432	10335.954
	0.017	6	0.016	0.015	14.179	10343.392
0.200	0.006	5	0.005	0.004	4.222	10240.097
	0.007	5	0.007	0.005	4.323	10251.403

	0.008	5	0.008	0.006	4.474	10261.436
	0.009	5	0.009	0.007	4.666	10271.429
	0.010	6	0.010	0.008	4.954	10281.325
	0.011	6	0.010	0.009	5.491	10290.973
	0.012	6	0.011	0.010	6.028	10300.620
	0.013	5	0.012	0.011	6.630	10310.202
	0.014	5	0.013	0.012	7.438	10319.579
	0.015	5	0.014	0.013	8.246	10328.956
	0.016	6	0.015	0.014	9.944	10337.442
	0.017	5	0.016	0.015	11.814	10345.756
	0.018	5	0.017	0.016	14.410	10353.348
0.250	0.007	4	0.007	0.006	3.143	10256.147
	0.008	4	0.008	0.006	3.202	10262.708
	0.009	4	0.009	0.007	3.292	10272.803
	0.010	4	0.010	0.008	3.458	10282.821
	0.011	4	0.011	0.009	3.650	10292.814
	0.012	4	0.012	0.010	3.841	10302.807

As in first stage, we analyzed the performance of portfolios generated by the model during the second stage by considering the portfolio net return. Although it is obvious that portfolio performance calculated using the equation (16) will differ according to the risk measure chosen (MAD in this case), some interesting results emerged as shown in Table 5.

$$\text{Portfolio performance} = \frac{\text{Mean return}}{\text{risk}} \tag{16}$$

It is evident that as the portfolio gross mean return increasingly vary for each diversification limit, the portfolio performance is on a decreasing trend.

Table 5. Stage (2) Optimal Portfolios' Performances

D.L	Net mean	Risk(MAD)	Performance	Cost
0.175	0.005	0.003	1.67	5.402
	0.005	0.004	1.25	5.479
	0.006	0.005	1.20	5.623
	0.007	0.006	1.17	5.814
	0.008	0.007	1.14	6.010
	0.009	0.008	1.13	6.210

D.L	Net mean	Risk(MAD)	Performance	Cost
0.175	0.010	0.009	1.11	6.611
	0.011	0.010	1.10	7.308
	0.012	0.011	1.09	8.036
	0.013	0.012	1.08	8.837
	0.014	0.013	1.08	9.644
	0.015	0.014	1.07	11.432
	0.016	0.015	1.07	14.179
0.200	0.005	0.004	1.25	4.222
	0.007	0.005	1.40	4.323
	0.008	0.006	1.33	4.474
	0.009	0.007	1.29	4.666
	0.010	0.008	1.25	4.954
	0.010	0.009	1.11	5.491
	0.011	0.010	1.10	6.028
	0.012	0.011	1.09	6.630
	0.013	0.012	1.08	7.438

D.L	Net mean	Risk(MAD)	Performance	Cost
	0.014	0.013	1.08	8.246
	0.015	0.014	1.07	9.944
	0.016	0.015	1.07	11.81
	0.017	0.016	1.06	14.41
0.250	0.007	0.006	1.17	3.143
	0.008	0.006	1.33	3.202
	0.009	0.007	1.29	3.292
	0.010	0.008	1.25	3.458
	0.011	0.009	1.22	3.650
	0.012	0.010	1.20	3.841
	0.013	0.011	1.18	4.763
	0.013	0.012	1.08	5.813
	0.014	0.013	1.08	6.863
	0.015	0.014	1.07	7.914
	0.016	0.015	1.07	9.013
	0.017	0.016	1.06	11.424
	0.018	0.017	1.06	14.709
	0.018	0.018	1.00	19.961

D.L	Net mean	Risk(MAD)	Performance	Cost
0.300	0.009	0.007	1.29	2.728
	0.010	0.008	1.25	2.840
	0.011	0.009	1.22	2.945
	0.012	0.010	1.20	3.128
	0.013	0.011	1.18	3.441
	0.014	0.012	1.17	4.450
	0.014	0.013	1.08	5.504
	0.015	0.014	1.07	6.737
	0.016	0.015	1.07	7.970
	0.017	0.016	1.06	9.202
	0.018	0.017	1.06	12.072
	0.018	0.018	1.00	16.109

3.2 Sensitivity Analysis

We perform sensitivity analysis of the SMADTC model by calculating sensitivity index (SI) for each parameter. This is achieved by finding the output percentage difference when varying one input parameter from its minimum value to its maximum value (Hoffman and Gardner, [11]; Bauer and Hamby,

[1]). We use the formula

$$\text{Sensitivity Index} = \frac{D_{\max} - D_{\min}}{D_{\max}}$$

where $D_{\min}$ and $D_{\max}$ are the minimum and maximum output values respectively, resulting from varying the input parameter over its entire range. This figure-of-merit provides a good indication of parameter and model variability (Hamby D. M., [8]). We take the minimum input value to be zero and let the maximum input value vary from 0.1 to 0.35, as shown in Table 6. The phrase 'max value' refers to 'maximum input parameter value' and it gives the maximum input of the parameter considered. We calculate the sensitivity index corresponding to each asset weight.The result in the table reflect small percentages of model variability due to each input change of each parameter. This shows that the output values are not strongly reliant on certain individual parameters. Hence, the SMADTC model shows to be a reliable portfolio optimization model.

However, when doing sensitivity analysis, the following is noted. The SMADTC model is stochastic and works by replacing one parameter by a less profitable one when we take a weight of an asset to be zero, thereby achieving a less 'optimal portfolio'. Here, a less preferable asset is included in the 'less optimal portfolio'. This is so because of a condition in the model that ensures that the total weight of assets in a portfolio be unit. The weight x_i is for asset $A_i, i = 1, \cdots, 13$. This, however, leads to relative sensitivity analysis. Thus, the $D_{\min}$ of model output is a relative value. Therefore, we perform the relative sensitivity analysis of the model by finding the difference between the optimal portfolio value when the asset under consideration attains maximum value and when it gets a value of zero (*i.e.*, excluded).

Table 6.SMADTC model Sensitivity Analysis

Max value	Parameter	Cost SI	Wealth SI
0.1	x_1	0.1618	0.0019
	x_2	-0.0607	0.0017
	x_3	0.0547	0.0016
	x_4	0.1534	-0.00006
	x_6	0.1348	-0.00005
	x_7	0.1078	0.0016
	x_9	0.0647	0.0004
	x_{10}	0.0377	0.0012
	x_{12}	0.1348	0.0010
	x_{13}	0.1240	0.0016

Max value	Parameter	Cost SI	Wealth SI
0.15	x_2	-0.0125	0.0052
	x_3	0.1281	0.0035
	x_7	0.2135	0.0028
	x_9	0.1281	0.0006
	x_{10}	0.0747	0.0017
	x_{12}	0.2598	0.0018
	x_{13}	0.2456	0.0029
0.2	x_2	-0.1065	0.0064
	x_3	0.2222	0.0039
	x_7	0.3704	0.0031
	x_9	0.2222	0.0074
	x_{10}	0.1296	0.0023
0.25	x_2	-0.2153	0.0071
	x_3	0.3529	0.0049
	x_9	0.3529	0.0009
	x_{10}	0.2059	0.0029

Max value	Parameter	Cost SI	Wealth SI
0.3	x_2	-0.1859	0.0101
	x_3	0.4615	0.0059
	x_9	0.4615	0.0043
	x_{10}	0.2692	0.0034

We also carry out sensitivity analysis of the model in which we allow the gross portfolio mean return to vary by gradually increasing it and analysing changes in optimal portfolio asset weights. The results are shown in Table 7. The abbreviation ‘A.weight’ means ‘Asset weight’. We observe that the model identifies assets with better returns but which have more implicit transaction costs whenever there is a change in the initial asset’s weight. However, such changes are minimal.

Table 7.Percentage portfolio compositions: Portfolio mean return constrained.

D.Lim	A.weight	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.030
0.1	x_1	10	10	10	10	10	10	10	10	10
	x_2	10	10	10	10	10	10	10	10	10
	x_3	10	10	10	10	10	10	10	10	10
	x_4	10	2.3	4.9	4.0					
	x_5		7.7	10	10	10	10	10	10	10
	x_6	10	10	5.1						
	x_7	10	10	10	10	10	10	10	10	10
	x_8								1.5	10
	x_9	10	10	10	10	10	10	10		
	x_{10}	10	10	10	10	10	10	10	8.5	
	x_{11}				6.0	10	10	10	10	10
	x_{12}	10	10	10	10	10	10	10	10	10
	x_{13}	10	10	10	10	10	10	10	10	10

D.Lim	A.weight	0.019	0.020	0.022	0.024	0.026	0.028	0.030	0.032	0.034
0.2	x_1					20	14.1	20	20	20
	x_2	20	20	20	20	20	20	20	20	20
	x_3	20	20	20	20	20	20	20	20	20
	x_5								8.5	15.2
	x_7	20	20	20	20		5.9			
	x_9	20	11.9							
	x_{10}	20	20	20	20	20	20	20	11.5	
	x_{11}			2.6	13.3	20	20	20	20	20
	x_{12}									4.8
	x_{13}		8.1	17.4	6.7					
		0.023	0.024	0.026	0.028	0.030	0.032	0.034	0.036	0.038
0.3	x_1						4.1	10.2	18.2	26.3
	x_2	30	30	30	30	30	30	30	30	30
	x_3	30	30	30	30	30	30	29.8	21.8	13.7
	x_7		10							
	x_9	10								
	x_{10}	30	30	29.7	21.2	12.7	5.9			
	x_{11}				10.3	18.8	27.3	30	30	30

4 Conclusion

In this study, a multi-stage stochastic transaction cost model is proposed which uses mean absolute deviation as its risk and takes into account uncertainty of assets' returns and of implicit transaction costs. The main contributions of this study include (*i*) the development of a stochastic trading cost model that generates optimal portfolios while minimizing uncertain implicit transaction costs incurred during initial trading and in subsequent rebalancing of the portfolios, (*ii*) the development of a strategy that captures uncertainty in stock prices and in corresponding implicit transaction costs by way of scenarios. To provide investors with competitive portfolio returns, investment managers must manage transaction costs pro-actively, because lower transaction costs mean higher portfolio returns. Ineffective transaction cost management may be very damaging in today's competitive environment, in which the difference between success and failure may be no more than a few basis points. The model has shown that it generates optimal portfolios whose net wealths are better that those generated by MV, MM, and MAD models. It also has the advantage of generating a least cost optimal portfolio as compared to the other three models. The methodology allows investors or investment managers to choose optimal portfolios with acceptable implicit cost levels to be incurred while executing their trading. It is a suitable strategy for conservative investors. The methodology is a linear programming model and hence reduces considerably the time needed to reach a solution. It is feasible for large-scale portfolio selection. It is left, however, for future research to develop a model that accounts for both implicit and explicit trading costs while considering uncertainty of both asset prices and transaction costs. The study, like any other, has its own limitations. The use of an asset's closing price for each month in the calculation of an asset's return rate may not be the most accurate measure. However, since all assets' rates of returns have been obtained in the same way, this does not prejudice the findings.

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5 Appendix

We present models and data sets used in the validation of the SMADTC model.

5.1 Mean-variance model: A1

The mean-variance (MV) model is proposed by Harry Markowitz [17], in which the portfolio that minimizes the variance subject to the restriction of a given mean return is chosen as the optimum portfolio. The mathematical model proposed by Markowitz is as follows:

Minimize

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{1}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} r_i x_i &\geq \rho, \\ \sum_{i=1}^{n} x_i &= 1, \\ 0 \leq x_i &\leq u_i, i = 1, \cdots, n. \end{aligned}$$

where σ_{ij} is the covariance between assets i and j, x_i is the proportion of wealth invested in asset i, r_i is the expected return of asset i in each period, ρ is the minimum rate of return desired by an investor, and u_i is the maximum proportion of wealth which can be invested in asset i.

5.2 Minimax model: A2

Young [21] propose the minimax (MM) model using minimum return as as measure of risk. This minimax model is equivalent to the mean-variance (MV) model if assets' returns are multivariate normally distributed. It is a linear programming model and is given as follows:

Maximize

$$M_p \tag{2}$$

subject to

$$\begin{aligned}
\sum_{i=1}^{n} w_i y_{it} - M_p &\geq 0, t = 1, \cdots, T \\
\sum_{i=1}^{n} w_i \bar{y}_i &\geq G, \\
\sum_{i=1}^{n} w_i &\leq W, \\
w_i &\geq 0, i = 1, \cdots, n.
\end{aligned}$$

where y_{it} is the return of one dollar invested in security i in time period t, $\bar{y}_i$ is the mean return of security i, w_i is the portfolio allocation to security i, M_p is the minimum return on the portfolio, G is the minimum level of return, and W is the total allocation.

5.3 Mean absolute deviation model: A3

Konno and Yamazaki [14] propose the L_1 risk function (mean absolute deviation - MAD) and show that it behaves in the same manner as the mean - variance model of Harry Markowitz [17] when the assets' returns are multivariate normally distributed. They develop the MAD model as given below:

$$H(x) = \frac{1}{T}\sum_{t=1}^{T}|\sum_{i=1}^{n}(r_{it} - r_i)x_i| \qquad (3)$$

subject to

$$\begin{aligned}
\sum_{i=1}^{n} r_i x_i &\geq \rho, \\
\sum_{i=1}^{n} x_i &= 1, \\
0 \leq x_i &\leq u_i, i = 1, \cdots, n.
\end{aligned}$$

where r_{it} is the realized return of asset i of period t, r_i is the expected return of asset i per period, and x_i is the proportion of wealth invested in asset i.

5.4 Data of assets

Table 8. Meaning of symbols

Symbol	A_1	A_2	A_3	A_4	A_5	A_6	A_7
Asset	AVI	ASR	APN	CSB	CLS	CML	MPC
Symbol	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	
Asset	PNC	SPP	TRU	CPI	IPL	WHL	

Table 9. Mean asset returns

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.099	0.422	-0.028	0.067	0.124	0.127	0.045
2	-0.102	-0.036	0.036	0.041	-0.057	-0.074	-0.078
3	-0.051	0.297	-0.027	0.143	-0.017	0.056	-0.082
4	-0.027	0.038	0.057	-0.021	0.014	-0.152	-0.042
5	-0.112	-0.036	-0.031	-0.138	-0.122	-0.188	-0.063
6	0.093	-0.125	0.32	0.036	0.062	0.099	0.304
7	0.084	0.071	0.113	0	0.179	0.04	0.134
8	-0.026	-0.193	-0.091	0.1	0	0.106	0.003
9	0.029	-0.173	-0.2	-0.018	0.03	-0.13	0.079
10	0.225	-0.2	0.101	-0.074	-0.048	0	-0.003
11	0.116	0.213	-0.103	0.04	0.118	-0.076	0.034
12	-0.074	-0.186	0.26	0.096	-0.029	-0.026	0.051
13	0.026	-0.139	-0.02	-0.035	-0.106	-0.022	-0.085
14	-0.136	0.294	-0.099	0.236	0.017	0.114	0.019
15	-0.046	0.02	-0.053	-0.043	0.097	0.163	0.076
16	0.061	0.069	0.111	0.022	0.064	-0.075	0.04
17	-0.029	0	0.141	-0.038	0.049	0.159	0.035

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
18	0.117	0.104	0.087	0.078	0.116	0.113	0.087
19	0.001	0.102	-0.017	0.014	-0.009	0.118	0
20	0	0.079	0.057	0.081	0.128	0.013	0.114
21	0.097	0.029	0.07	-0.077	0.079	0.048	0.056
22	-0.074	0.08	0.027	0.017	0.018	0.035	-0.092
23	0.083	0.004	0.082	0.056	0.083	0.048	0.074
24	0.012	-0.011	-0.084	-0.027	-0.028	0.017	0.005
25	0.033	0.036	0.053	-0.014	0.042	-0.006	0.095
26	0.089	0.069	0.12	0.034	0.074	0.101	0.032
27	0.008	0.018	0.05	-0.011	0.044	0.176	0.053
28	-0.031	-0.133	-0.052	-0.05	0.051	-0.07	0.069
29	-0.049	0.015	-0.039	0.072	0.043	0.028	-0.001
30	0.131	0.112	0.071	-0.027	0.065	0.16	0.128
31	0	-0.091	0.032	-0.068	0.037	0.004	-0.022
32	0.088	0.117	0.118	0.147	0.179	0.16	0.113

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
33	0.029	0.077	-0.005	0.212	0.034	0.014	0.157
34	0	0	-0.005	-0.049	-0.049	0.125	0.009
35	0.076	0.125	-0.011	0.057	-0.003	0.111	0.036
36	-0.005	0.055	-0.074	-0.048	-0.085	-0.069	-0.145
37	0.01	0.028	-0.043	-0.074	-0.029	-0.052	0.054
38	-0.01	0.056	-0.031	0.116	0.105	0.076	0.022
39	0.027	-0.012	0.025	-0.005	0.031	0.067	0.097
40	-0.028	0.012	0.077	0.043	-0.025	0.034	-0.047
41	0.042	0.014	-0.037	-0.023	-0.013	-0.018	0.065
42	0.033	0.02	-0.008	-0.035	-0.053	0.037	0.081
43	0.014	-0.018	0.014	0.011	0.043	-0.006	0.001
44	-0.011	-0.105	0.08	0.113	-0.097	0.005	-0.088
45	0.101	0.099	0.044	0.001	0.107	0.128	0.135

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
46	0.049	0.012	0.023	0.086	0.068	0.007	0.047
47	0.063	-0.041	-0.008	0.055	0.039	0	-0.003
48	0.045	0.086	0.031	-0.005	-0.141	0.117	0.081
49	0.057	0.106	0.09	-0.053	0.106	0.083	0.034
50	0.057	-0.043	0.091	0.133	0.019	0.038	0.057
51	0.039	0.098	0.061	0.04	0.046	0.025	0.115
52	0.01	0.057	-0.065	-0.007	0	-0.099	-0.01
53	0.027	0.068	0.07	0.054	0.149	0.051	0.074
54	0.182	-0.01	0.153	0.102	0.016	0.064	0.101
55	-0.009	0.007	-0.008	0.053	-0.003	0.02	0.049

Table 9. Mean asset returns (continued)

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.136	-0.008	0.02	0.177	0.053	0.057
2	0	-0.025	-0.038	-0.064	-0.009	-0.025
3	-0.022	0.155	0.004	-0.096	-0.034	-0.013
4	-0.068	-0.071	-0.105	-0.076	-0.308	-0.054
5	-0.146	-0.048	0.002	-0.115	-0.005	-0.097
6	-0.057	0.051	0.267	0.107	-0.165	0.155
7	0.288	0.038	0.073	-0.013	0.226	0.067
8	-0.141	-0.065	-0.064	0.085	0.094	-0.069
9	-0.285	0.129	0.136	-0.156	-0.07	-0.034
10	-0.272	-0.056	-0.043	0.037	-0.084	0.084
11	-0.042	0.06	0.075	0.036	0.184	0.038
12	0.049	-0.035	0.033	0.034	-0.137	0.058
13	-0.063	-0.031	-0.117	0	-0.138	-0.115
14	0.039	-0.026	0.032	0.081	0.207	-0.042
15	0.022	0.026	0.058	0.202	0.02	0.053
16	0.089	0.07	0.068	0.051	0.101	0.042
17	0.097	-0.004	0.019	0.035	-0.028	0.036

Month	PNC	SPP	TRU	CPI	IPL	WHL
18	0	0.054	0.081	0.131	0.167	0.202
19	-0.004	0.043	-0.016	0.146	0.09	0.012
20	0.456	0.037	0.08	0.109	0.077	0.025
21	-0.033	0.066	0.059	0.049	0.02	0.083
22	-0.041	-0.041	-0.075	0.109	-0.006	-0.048
23	-0.02	0.074	0.048	0.111	0.092	0.072
24	0.1	-0.001	-0.028	-0.024	-0.092	0.024
25	0.125	0.028	0.166	0.065	0.17	0.127
26	0.078	0.021	0.058	0.153	0.07	0.093
27	-0.023	0.029	0.017	0.076	-0.018	0.041
28	0.103	0.019	0.022	-0.03	-0.025	-0.013
29	-0.002	0.022	-0.013	0.197	-0.107	0.035
30	0.131	0.071	0.087	0.093	0.124	0.085
31	-0.021	-0.017	0.004	0.035	0.059	-0.053
32	0.114	0.105	0.196	0.116	0.111	0.099
33	0.068	0.025	-0.014	-0.021	0.007	0.015
34	0.092	0.062	0.043	0.019	0.072	-0.038
35	0.107	-0.033	-0.005	0.156	0.041	0.02

Month	PNC	SPP	TRU	CPI	IPL	WHL
36	-0.034	-0.071	-0.11	-0.109	-0.137	-0.128
37	0.03	0.046	0	0.039	0.021	0.136
38	0.029	0	0.106	0.057	0.016	0.047
39	0.014	0.026	0.078	0.037	0.033	0.072
40	0.099	-0.05	-0.047	0.085	-0.013	-0.007
41	0.154	-0.026	0.012	-0.042	0.041	0
42	0.022	0.02	-0.012	0.001	-0.052	0.061
43	-0.076	0.035	0.071	0.04	0.007	0.155
44	0.116	0.011	-0.091	0.019	-0.091	-0.04
45	0.138	0.007	0.138	-0.052	0.116	0.154
46	-0.029	0.16	-0.012	0.011	0.007	0.001
47	0.044	-0.035	-0.068	-0.033	0.044	-0.035
48	0.185	0.02	0.059	0.024	0.122	0.077
49	0.059	0.041	0.034	0.015	0.063	0.074
50	0.039	0.004	-0.001	0.108	0.052	0.067
51	0.113	0.05	0.028	0.083	0.089	0.01
52	-0.037	-0.097	-0.009	0.017	-0.021	0.003
53	0.035	0.035	0.089	-0.058	0.04	0.031

Month	PNC	SPP	TRU	CPI	IPL	WHL
54	0.002	0.054	0.154	0.026	0.1	0.071
55	0.061	0.011	0.003	-0.012	0.005	0.016

Table 10.Asset Transaction Cost rates

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
1	0.0030	0.0023	0.0028	0.0036	0.0050	0.0119	0.0073
2	0.0004	0.0009	0.0028	0.0029	0.0023	0.0061	0.0051
3	0.0054	0.0008	0.0022	0.0052	0.0032	0.0425	0.0088
4	0.0087	0.0064	0.0121	0.0040	0.0107	0.0011	0.0019
5	0.0043	0.0062	0.0027	0.0080	0.0034	0.0004	0.0229
6	0.0000	0.0140	0.0072	0.0386	0.0098	0.0067	0.0012
7	0.0066	0.0048	0.0195	0.0214	0.0147	0.0018	0.0128
8	0.0048	0.0044	0.0306	0.0171	0.0548	0.0137	0.0024
9	0.0042	0.0061	0.0063	0.0078	0.0132	0.0018	0.0007
10	0.0100	0.0220	0.0027	0.0018	0.0213	0.0025	0.0060
11	0.0060	0.0003	0.0112	0.0258	0.0041	0.0085	0.0136
12	0.0143	0.0045	0.0170	0.0049	0.0015	0.0258	0.0019
13	0.0003	0.0126	0.0010	0.0055	0.0083	0.0135	0.0037
14	0.0033	0.0051	0.0029	0.0041	0.0019	0.0015	0.0020
15	0.0064	0.0023	0.0004	0.0001	0.0228	0.0026	0.0082
16	0.0057	0.0114	0.0042	0.0054	0.0092	0.0139	0.0140
17	0.0010	0.0087	0.0435	0.0165	0.0099	0.0073	0.0105

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
18	0.0537	0.0047	0.0067	0.0212	0.0045	0.0109	0.0028
19	0.0007	0.0052	0.0027	0.0474	0.0056	0.0053	0.0113
20	0.0060	0.0117	0.0126	0.0856	0.0046	0.0295	0.0033
21	0.0061	0.0303	0.0052	0.0532	0.0169	0.0033	0.0127
22	0.0070	0.0003	0.0087	0.0194	0.0268	0.0149	0.0049
23	0.0076	0.0029	0.0046	0.0074	0.0003	0.0008	0.0103
24	0.0056	0.0473	0.0025	0.0069	0.0127	0.0055	0.0026
25	0.0018	0.0058	0.0095	0.0070	0.0079	0.0046	0.0056
26	0.0149	0.0074	0.0327	0.0072	0.0009	0.0019	0.0045
27	0.0037	0.0208	0.0032	0.0233	0.0104	0.0035	0.0165
28	0.0202	0.0065	0.0256	0.0301	0.0051	0.0021	0.0073
29	0.0068	0.0225	0.0141	0.0061	0.0018	0.0056	0.0116
30	0.0298	0.0000	0.0343	0.0068	0.0695	0.0253	0.0094
31	0.0072	0.0197	0.0027	0.0408	0.0084	2.0000	0.0115
32	0.0157	0.0153	0.0066	0.0679	0.0040	0.0749	0.0058
33	0.0139	0.0238	0.0015	0.0019	0.0040	0.0286	0.0111
34	0.0142	0.0024	0.0079	0.0073	0.0013	0.0105	0.0024
35	0.0074	0.0186	0.0068	0.0326	0.0198	0.0066	0.0114

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
36	0.0316	2.0000	0.0082	0.0146	0.0487	0.0074	0.0133
37	0.0041	0.0083	0.0433	0.0155	0.0369	0.0082	0.0075
38	0.0459	2.0000	0.0058	0.0014	0.0029	0.0354	0.0033
39	0.0006	0.0386	0.0012	0.0046	0.0334	0.0177	0.0339
40	0.0743	0.0225	0.0097	0.0044	0.0148	0.0202	0.0543
41	0.0210	0.0290	0.0103	0.0952	0.0144	0.1277	0.0402
42	0.0208	0.0022	0.0179	0.0222	0.0308	0.1053	0.0089
43	0.0217	2.0000	0.0177	0.0396	0.0238	2.0000	0.0052
44	0.0053	2.0000	0.0151	0.0619	0.0033	0.0942	0.0112
45	0.0124	0.0942	0.0077	0.0759	0.0183	0.0408	0.0083
46	0.0086	0.0083	0.0800	0.0352	0.0215	0.0258	0.0014
47	0.0169	0.0645	0.0026	0.0829	0.0064	0.0077	0.0271
48	0.0060	0.0072	0.0185	0.0769	0.0068	0.0198	0.0020
49	0.0209	2.0000	0.0255	0.1452	0.0000	0.0217	0.0047
50	0.0330	0.0108	0.0132	0.1125	0.0043	0.0089	0.0277
51	0.0040	0.0189	0.0016	0.0894	0.0086	0.0308	0.0114
52	0.0032	0.0028	0.0110	2.0000	0.0078	0.0048	0.0164

Month	AVI	ASR	APN	CSB	CLS	CML	MPC
53	0.0317	0.0237	0.0026	0.0734	0.0007	0.0543	0.0101
54	0.0180	0.0220	0.0085	0.0112	0.0104	0.0151	0.0228

Table 10. (continued)

Month	PNC	SPP	TRU	CPI	IPL	WHL
1	0.0319	0.0019	0.0009	0.0063	0.0058	0.0062
2	0.0323	0.0072	0.0033	0.0105	0.0132	0.0010
3	0.0038	0.0044	0.0241	0.0061	0.0067	0.0018
4	0.0049	0.0035	0.0028	0.0411	0.0040	0.0029
5	0.0058	0.0021	0.0031	0.0031	0.0005	0.0013
6	0.0046	0.0024	0.0056	0.0031	0.0050	0.0010
7	0.0315	0.0029	0.0066	0.0056	0.0050	0.0008
8	0.0086	0.0946	0.0540	0.0121	0.0542	0.0069
9	0.0263	0.0009	0.0068	0.0003	0.0019	0.0027
10	0.0148	0.0030	0.0128	0.0108	0.0052	0.0023
11	0.0202	0.0019	0.0063	0.0459	0.0121	0.0689
12	0.0043	0.0109	0.0029	0.0071	0.0078	0.0099
13	0.0067	0.0007	0.0023	0.0028	0.0042	0.0044
14	0.0127	0.0017	0.0033	0.0032	0.0043	0.0057

Month	PNC	SPP	TRU	CPI	IPL	WHL
15	0.0197	0.0041	0.0036	0.0191	0.0084	0.0020
16	0.0072	0.0012	0.0177	0.0047	0.0100	0.0050
17	0.0074	0.0001	0.0185	0.0050	0.0008	0.0004
18	0.0122	0.0085	0.0077	0.0005	0.0002	0.0098
19	0.0088	0.0007	0.0015	0.0054	0.0085	0.0015
20	0.0081	0.0078	0.0324	0.0030	0.0027	0.0046
21	0.0179	0.0085	0.0022	0.0031	0.0037	0.0271
22	0.0228	0.0022	0.0011	0.0101	0.0097	0.0234
23	0.0106	0.0305	0.0077	0.0143	0.0010	0.0220
24	0.0293	0.0053	0.0024	0.0039	0.0137	0.0073
25	0.0212	0.0050	0.0127	0.0125	0.0205	0.0421
26	0.0452	0.0019	0.0145	0.0145	0.0034	0.0121
27	0.0178	0.0024	0.0137	0.0069	0.0047	0.0223
28	0.0203	0.0093	0.0027	0.0053	0.0064	0.0071
29	0.0526	0.0043	0.0026	0.0037	0.0040	0.0019
30	0.0595	0.0113	0.0095	0.0065	0.0133	0.0016
31	2.0000	0.0058	0.0069	0.0127	0.0096	0.0028
32	0.0470	0.0020	0.0021	0.0140	0.0086	0.0300

Month	PNC	SPP	TRU	CPI	IPL	WHL
33	0.0287	0.0151	0.0018	0.0047	0.0047	0.0114
34	0.0030	0.0023	0.0068	0.0149	0.0068	0.0025
35	0.0545	0.0032	0.0261	0.0091	0.0125	0.0184
36	0.0590	0.0017	0.0119	0.0092	0.0103	0.0032
37	0.0874	0.0142	0.0117	0.0012	0.0022	0.0117
38	0.0597	0.0039	0.0055	0.0241	0.0013	0.0080
39	0.2246	0.0186	0.0064	0.0102	0.0215	0.0058
40	0.0488	0.0107	0.0041	0.0015	0.0079	0.0302
41	0.0845	0.0252	0.1433	0.0481	0.0299	0.1060
42	0.0207	0.0077	0.0088	0.0236	0.0164	0.0105
43	0.2118	0.0357	0.0239	2.0000	0.0077	0.0008
44	0.0267	0.0080	0.0066	0.0142	0.0078	0.0123
45	0.2449	0.0741	0.0075	0.0714	0.0082	0.0370
46	0.2290	0.0050	0.0187	0.0546	0.0328	0.0481
47	0.0047	0.0093	0.0123	0.0165	0.0110	0.0169
48	0.0625	0.0012	0.0007	0.0479	0.0099	0.0273
49	0.1374	0.0228	0.0110	0.0147	0.0042	0.0069
50	0.0247	0.0179	0.0013	0.0165	0.0024	0.0608

Month	PNC	SPP	TRU	CPI	IPL	WHL
51	0.0114	0.0011	0.0155	0.0469	0.0007	0.0109
52	0.0270	0.0062	0.0650	0.0408	0.0336	0.0198
53	0.0110	0.0082	0.0633	0.0524	0.0507	0.0112
54	0.0228	0.0445	0.0136	0.0202	0.0033	0.0347