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Pedro de Mendonça

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EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 298 8491
Fax: +322 298 8490
www.eeri.eu

Nonlinear Phenomena in a Growing Economy with Convex Adjustment Costs☆

Pedro de Mendonça[1]

Abstract

We discuss the implications of nonlinear dynamic phenomena for policy definition in a growing economy. Departing from the hypothesis that local analysis of economic systems focus the definition of policy on short run outcomes, we propose that to have a clearer perspective on long run outcomes, we have to focus the analysis on the study of local bifurcations and global dynamics instead. This approach, in our opinion, provides a better insight on complex macroeconomic phenomena and allows for a better definition of policy rules in a long run horizon. To demonstrate this hypothesis, we set up a representative agent economy based on neoclassical assumptions, where agents face convex risk premium and investment adjustment costs. This economy describes an endogenous optimal growth model, that has solutions given by a nonlinear three-dimensional dynamical system. To evaluate this system, we resort to qualitative analysis methods and show the existence of fold (saddle-node), hopf and fold-hopf bifurcations, in a multiple equilibria environment. Numerical results suggest the absence of local stable solutions for a wide range of parameter values. We then focus our analysis on the complex organization of the economy phase-space and evaluate several conjectures related to the existence of complex nonlinear phenomena in the vicinity of fold-hopf bifurcation points. We relate these conjectures with the hypothesis of endogenous structural change and discuss the implications of complex global dynamic phenomena for long run policy definition.

Keywords: Nonlinear Growth Dynamics, Financial Instability, Fold-Hopf Bifurcation

1. Introduction

The aim of this paper is to discuss the implications of global nonlinear phenomena for long run economic policy definition. We start from the hypothesis that in order to get further insight on long run macroeconomic phenomena, we have to extend our knowledge on nonlinear economic dynamics and the underlying global scenarios. Our approach is based on the following argument. The focus on local dynamics of economic systems leads to a short run policy focus. Therefore, in order to improve the effectiveness of policy in longer horizons, we have to improve our knowledge of environments where global stability conditions no longer apply. To demonstrate this hypothesis, we propose a model of endogenous optimal growth based on simple and well known economic assumptions. Endogenous growth theory was introduced by the seminal proposals of Romer [74], Romer [75], Uzawa [83] and Lucas [52]. We depart from a deterministic intertemporal optimization framework, following the optimal growth neoclassical framework of Ramsey [70], Cass [16] and Koopmans [46], and set up this model as an open economy populated with *N* representative agents, assuming neoclassical market clearing micro foundations. Our framework is closely related to Romer [74] proposal, as the growth engine of this economy is also driven by linear productive capital growth. Agents solve an optimal control consumption/investment problem in continuous time, following the seminal proposals of Merton [59]. Although the investor problem has its roots on the field of financial mathematics, it is widely used for modeling open economies, given that on aggregate, the national income identity can be matched by the individual budget constraints[2]. We assume that agents in our

☆The routines and data used in this working paper can be downloaded here.

Email address: `g.p.a.de-mendonca@warwick.ac.uk` (Pedro de Mendonça)

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[2]Turnovsky [82] provides a interesting mathematical discussion in continuous time modeling for open economy macrodynamic setups.

economy face two nonlinear mechanisms, defined by convex risk premium on bonds and investment adjustment costs, following the well established proposals of Bardhan [7] and Hayashi [37], respectively. The paper by Eicher et al. [27] is a recent example of an economic growth setup closely related to ours that assumes the existence of these two nonlinearities. Our main objective is to evaluate the conditions for existence of global optimal growth dynamics. We follow a straightforward technical analysis of our problem, based on local qualitative analysis, to show the existence of nonlinear phenomena, such as *Hopf*, fold (saddle-node) and fold-hopf bifurcations, consistent with economic feasible scenarios. A thorough numerical exploration of the parameter space does not reveal the existence of local stable solutions, following the *Routh-Hurwitz* stability criterion. Given this result, we focus our analysis on the definition of scenarios consistent with the existence of asymptotic optimal dynamic stable solutions. This is a reasonable objective for policy in a complex dynamic setup, where stable long run dynamics depend on the interaction of multiple equilibrium solutions. In order to define scenarios consistent with this criterion, we discuss several conjectures consistent with local bifurcation phenomena and the complex organization of this economy phase-space. The existence of fold-hopf bifurcations suggests the existence of solutions driven by heteroclinic and homoclinic orbits. Both these scenarios have economic interpretation and are meaningful for policy purposes. We relate these conjectures to the hypothesis of endogenous structural change. We then extend global analysis of this system and discuss the conditions for the existence of natural frontiers of the economic space, in the form of separatrix planes arising from the dynamics in the vicinity of the non-meaningful set of steady-states. This analysis suggests that the study of non feasible solutions in nonlinear economic models may provide meaningful insight for policy, in particular for economies facing dire institutional conditions.

Our proposal departs from Richard Goodwin's main paradigm. Goodwin considered that the extreme phenomena observed in economic data could only be explained by nonlinearity. Goodwin's seminal proposals, such as the nonlinear accelerator model, Goodwin [31][3], are still the main benchmarks of evolutionary economic dynamics theory. Unfortunately, Goodwin's innovative proposal was largely dismissed in mainstream macroeconomic theory, on the grounds that the model's main nonlinear mechanism, a forced oscillator, had no justification in economic theory[4]. In order to avoid this criticism, we model our economy as an endogenous optimal growth model based on mainstream neoclassical assumptions and focus our analysis on the interpretation and evaluation of global dynamic phenomena. The discussion on global dynamic phenomena in theoretical growth economics has roots in mainstream literature that date back to the seventies decade. The focus then was on the definition of sufficient conditions guaranteeing global asymptotic stability. The papers by Brock and Scheinkman [15], Cass and Shell [17], Rockafellar [73] and Nishimura [62] remain some of the main proposals on this topic. Recent literature on global dynamic economics has focused on the existence of history dependence in nonlinear models of optimal growth with multiple equilibria. A thorough discussion on the mathematics central to many economic applications, along with a careful literature review on this topic can be found in Deissenberg et al. [24].

Although there is no absolute and universal approach to economic phenomena, radical thinking has been consistently deterred in economic research due to the established orthodox approaches. Change comes slowly in economics and usually involves a long process of reform. In many cases this particular process of evolution led to the dismissal of many interesting ideas. When some of these ideas are able to establish themselves in academia, it does not mean that they are taken in consideration in the development of economic policy agenda. To justify this argument, we put forward two examples of this process that are related to the broad topic of evolutionary economic dynamics. In a recent book, Kirman [44][5], provides evidence that the orthodox view on the *Marshallian* demand curve, as a microeconomics law, has been incorrectly extrapolated[6], from aggregate market data[7]. These polemics on the evolutionary nature of economic phenomena have older roots in economic philosophy. Already in the nineteenth century, Veblen [84] questions why economics is not an evolutionary science? Some decades later, the famous dispute between John Maynard Keynes and Franck Ramsey on the nature of probability in economic phenomena, paved the way for the introduction of the role of subjectivity in economic theory[8].

[3]The nonlinear accelerator model suggested the persistence of business cycles and periodic dynamics consistent with the long wave hypothesis. This conjecture was first put forward by the Russian economist Nicolai Kondratiev, in Kondratiev [45]. For a discussion on the theoretical implications of Kondratiev proposal see Louça [51].

[4]There is an economic argument that interprets the countercyclical role of government spending as a source of forced oscillations in the economy. For a detailed discussion on this topic see Chian [20].

[5]Chapter 3- Fish Markets: An Example of the Emergence of Aggregate Coordination

[6]A formal proof on the impossibility of deriving aggregate demand in markets with heteregeneous agents is given by Sonnenschein [78]. A further mathematical discussion on the topic of market demand and excess demand functions can be found in Shafer and Sonnenschein [76].

[7] Using qualitative and quantitative data on the Marseille fish market, Kirman develops a simple adaptive evolutionary model that can reasonably replicate aggregate market dynamics. The author shows that the demand law holds on the aggregate. However, this aggregate behaviour arises from the interaction of heterogeneous agents that do not always have a demand curve with a negative slope.

[8]Keynes argued that in the future it would be possible to define economic phenomena in an objective probabilistic fashion, following the developments on theoretical physics in the first decades of the past century. On the other hand, Ramsey believed that there exists a degree of subjectivity driving economic decisions and any probabilistic approach to economic phenomena would have to take into account this feature.

These initial discussions led to the development of game theory by John von Neumann and Oskar Morgenstern, which later influenced systems approach to social sciences and is presently one of the main paradigms in modern evolutionary theory. The concept of heterogeneous strategic behaviour under subjectivity, for example, is now a crucial paradigm in the field of financial economics. In two seminal papers, Brock and Hommes [13] and Brock and Hommes [14], show that adaptive evolutionary behaviour can arise in rational decision systems, where agents have heterogeneous beliefs (fundamentalists vs. chartists). The authors show the existence of homoclinic bifurcations and chaotic dynamics, arising as a consequence of adaptive beliefs. This hypothesis is considered a plausible justification for the existence of extreme events in financial markets, in particular exchange rate markets[9].

Our proposal draws from this last example and proposes to evaluate the implications of global dynamics in an endogenous growth framework with neoclassical assumptions. As previously described, the dynamics of this economy are determined by an autonomous nonlinear dynamical system in continuous time. According to Mackay [54] low dimensional nonlinear dynamics does not belong to the field of complex systems. We agree with this interpretation, in the sense that nonlinear low dimensional dynamics does not involve many interdependent components. However, we believe that the exploration of global dynamic scenarios that are consistent with optimal control solutions, and the interpretation of such outcomes in growth models, still lie on the field of complex problems. This is particularly true for growth models with dynamics described by vector fields in $\Re^3$. In nonlinear dynamics literature one can find several applications that illustrate the complex challenges posed by such systems. An example of a system with similar characteristics to the one proposed in this paper is the Rabinovitch-Fabrikant system, following the proposal by Rabinovich and Fabrikant [69]. Danca and Chen [23] perform an extensive analysis of this vector field in $\Re^3$ and show that the global analysis of systems with quadratic and cubic terms is not straightforward. The authors also show that classical numerical integration methods are not reliable in this context.

We argue that in nonlinear setups, it is crucial for policy definition to have a global perspective of the dynamics of a system. This argument is reinforced by the absence of stable economic feasible solutions. In this environment, asymptotic stable solutions are only possible when we consider the existence of complex dynamics driven by the interaction of multiple equilibria. This outcome suggests a trade-off between stability and complexity in our system. Further, the existence of solutions undergoing fold-hopf bifurcations suggests that several complex global dynamic conjectures with dramatic policy implications are possible. Chen [18] argues that this trade-off has to be considered, if we wish to study economic systems in a complex framework. The author also argues that the empirical evidence regarding endogenous structural change and nonlinear dynamics can only be tackled by evolutionary theory. The hypothesis of chronic macroeconomic instability has its roots in economics literature. Minsky [61] put forward the financial instability hypothesis and suggested that business cycles are driven by financial decisions. Minsky's financial instability proposal is rooted in the *Schumpeterian* evolutionary hypothesis, which attributed an important role to financial intermediation as a driver of the long run cycle. In a recent book, Reinhart and Rogoff [71] show that financial crisis and instability are a common feature in macroeconomic history. Unfortunately, Minsky was never able to translate his idea into a consistent mathematical dynamic setup[10]. The endogenous structural change hypothesis is supported by empirical data on real macroeconomic aggregates. Figure 1 shows evidence of structural change for US and UK $\log(GDP)$ data. What the modern growth literature has been unable to explain, are the mechanisms leading to structural change, depicted by the shifts in intercept and slope [11]. The reason for this shortfall on literature is, in our opinion, related to the systematic approach based on linear and quasi-linear dynamic optimization problems[12]. We firmly believe that the introduction of further nonlinearities in growth models may shed some light on the dynamics of structural change, which can be linked to the existence of global economic dynamic phenomena. The outcomes portrayed in Figure 1, for example, can be related to existence of heteroclinic and homoclinic dynamics, leading to permanent and temporary structural

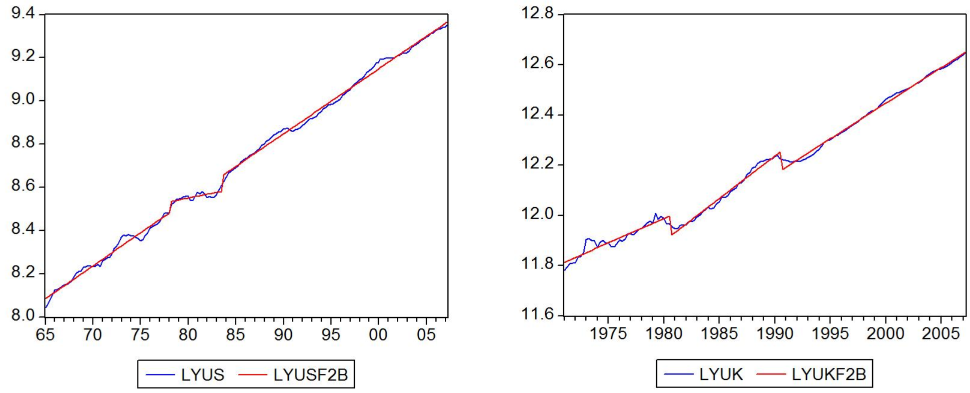
[9] A detailed discussion on this topic can be found in Hommes and Wagener [40].

[10] Keen [43] suggests that Minsky's failure to devise such setup was linked to the use of the multiplier-accelerator model as a setup for his proposal. The financial instability hypothesis has recently be regaining a renewed attention following the events surrounding the recent financial crisis. Recent discussions propose a reinterpretation of Minsky's original hypothesis closer to the evolutionary long cycle hypothesis. Palley [64] discusses the hypothesis of financial instability as a super-cycle. Keen [43] follows the same lines and proposes a redefinition of the original setup based on the Goodwin [32] nonlinear setup.

[11] The data fitting model used in here is based in the well known methodology developed by Vogelsang and Perron [86], Bai and Perron [5] and Bai and Perron [6] for estimation of multiple structural change events in data, following the structural change hypothesis by Perron [67]. This method is based on a consistent error minimizing estimator. We use the simple Additive-Outlier (A-O) Crash/Change specification on this data and define models with 1 to 4 statistically significant structural changes and estimate Crash models for unemployment data at estimated break dates. The final specifications are chosen using a ranking method based on several data fitting statistical indicators. Similar outcomes are observed on additional estimations using data from seventeen OECD economies.

[12] In earlier versions of this paper, we show that when only one nonlinearity is considered, the dynamic properties of these economies are consistent with the properties of linear systems.

change phenomena, respectively[13].



(a) US log GDP and fitted trend with 2 breaks

(b) UK log GDP and fitted trend with 2 breaks

Figure 1: Evidence of trend dynamics and structural change on log (GDP) data.

We organize our presentation in the following fashion. In section 2, we introduce the representative agent setup and the intertemporal maximization problem. In section 3, we show how the solution to the optimal control problem can be derived via an autonomous dynamical system, and derive sufficient conditions for the existence of optimal growing solutions. In section 4 and section 5, we put forward the main conditions describing local dynamics and local bifurcations. In section 6, we discuss the role of non feasible solutions and their implications for policy in a complex dynamic setup. Finally, in section 7, we introduce our main global dynamic conjectures, discuss their economic interpretation, and their implications for policy definition in a nonlinear environment.

2. The representative agent economy

We consider an open competitive economy populated by N representative agents (identical individuals) that live infinitely for $t \in [0, T]$, where $T = \infty$. Households invest in domestic and foreign capital in exchange for returns on these assets, purchase goods for consumption, and save by accumulating domestic and foreign assets. We further consider that agents can resort to debt accumulation to finance investment in domestic assets and/or consumption. Households may also undertake temporary disinvestment decisions on domestic capital to improve their financial balances. The representative agent seeks to maximize an intertemporal utility consumption function, $U(c)$, and discounts future consumption exponentially at a constant rate $\rho \in \Re^+$. To achieve this objective, agents solve an infinite horizon consumption, $c(t) \in \Re^+$, and investment, $i(t) \in \Re$, dynamic optimization problem *a la Merton*, taking into account the evolution of their budget constraint, $b(t) \in \Re$, and their domestic capital accumulation, $k(t) \in \Re^+$. The objective of each agent is to maximize the flow of discounted consumption outcomes,

$$U(c) = \int_0^T u(c(t))\, e^{-\rho t} dt, \text{ with } u(c(t)) = c(t)^\gamma, \tag{1}$$

where γ defines the intertemporal substitution elasticity in consumption, measuring the willingness to substitute consumption between different periods. A smaller γ means that the household is more willing to substitute consumption over time. We impose the usual constraint on the intertemporal substitution parameter, $0 < \gamma < 1$, such that $u'(c(t)) > 0$. This specification for utility belongs to the family of constant relative risk aversion (CRRA) utility functions and is widely used in optimization setups, where savings behaviour is crucial, such as economic growth problems. This setup also guarantees the concavity of the utility function, $u''(c(t)) < 0$. This is a necessary condition to obtain optimal solutions to our dynamic optimization problem as an initial value problem [14].

[13] For the UK case we estimate two breaks, 1980:4 and 1990:4, and aggregate change in intercept and slope equal to −0.158927 and 0.002192, respectively. For the US case we estimate two breaks, 1978:2 and 1983:4, and aggregate change in intercept and slope equal to −0.01222375 and −0.000117, respectively.

[14] We follow closely the seminal results of Arrow and Kurtz [3], Chapter 2- Methods of optimization over time, which guarantee the *Pontryagin* first order conditions are sufficient for determining an optimum solution in infinite horizon dynamic optimization problems with constant discount, provided that the objective function is concave and the transversality conditions are fulfilled. Turnovsky [81], chapter 3- Intertemporal optimization, discusses the definition of transversality conditions for two sector models, such as ours.

Following Barro and Sala-i-Martin [8], setups with infinitely lived households have the following interpretation. Each household contains more than one adult, defining the current generation population. In making plans, these adults take account of the welfare and resources of their prospective descendants. We model this intergenerational interaction by imagining that the current generation maximizes utility and incorporates a budget constraint over an infinite horizon. That is, although individuals have finite lives, we consider an immortal extended family.

This economy has N identical firms, owned by each household and producing an homogeneous good, $y(t) \in \Re^+$, that requires just capital inputs, $k(t)$. We assume this simplification for mathematical reasons, nevertheless, the domestic capital stock can be considered as a broad measure of available capital in the economy used in the production of goods. The technology level of firms is identical and given exogenously by parameter A. We do not consider the possibility of technological progress in this economy. The flow of output produced by each firm, at a given period, is given by a AK production function, expressed by equation (2), following the simple Romer [74] endogenous growth framework with marginal and average product constant at the level $A \in \Re^+$,

$$y(t) = Ak(t). \tag{2}$$

As usual in open economy frameworks, we assume that agents and firms have full access to international capital markets. Households can accumulate foreign debt/assets, $b(t)$, for which they pay/receive an exogenous interest, expressed in terms of the real international interest rate, r, plus a risk premium defined by the evolution of their real financial balances ratio, $b(t)/k(t)$. We assume that foreign debt payments, $b(t) > 0$, and returns on foreign assets, $b(t) < 0$, follow a convex specification, $\Xi(t)$, where, $\Xi(t)'_{b(t)} > 0$ and $\Xi(t)''_{b(t)} > 0$, for $b(t) > 0$. This specification follows closely the original proposal by Bardhan [7][15]. Interest payments are defined by

$$\Xi(t) = rb(t)\left(1 + \frac{d}{2}\frac{b(t)}{k(t)}\right), \tag{3}$$

where parameter $d \in \Re$ stands for the exogenous institutional risk premium, resulting from international capital markets sentiments on the quality of the debt bonds issued by the economy. This assumption is justified by bias arising from historical and psychological perceptions. We assume that macroeconomic factors are priced in the risk premium valuation through the net foreign assets to domestic capital ratio. A higher value of d means that holding the country debt bonds yields a higher risk for international investors, but investment by nationals on foreign assets pays a greater premium. A smaller value of d means that holding the country debt bonds yields a small risk for international investors, but investment by nationals on foreign assets pays a smaller premium. This setup can be interpreted in terms of the degree of development and international financial integration of a given economy. International investors' sentiment towards a mature economy is less severe, as a consequence of the higher degree of international trade and financial integration. This phenomenon can be explained by historical, political and economic factors, which bias international investors sentiments towards *successful* economies, while disregarding real economic information. It can also result from information costs, which deter international investors from acquiring relevant information on the state of a specific economy and increases investors reliance on individual or collective market beliefs. A smaller d represents also a smaller premium for residents investing in foreign assets. This can be interpreted as a result of the higher degree of international financial integration in mature economies. Residents of developed economies require smaller premiums on their foreign investments due to smaller transaction and information costs of investing abroad, arising from financial innovation in developed economies banking systems. Therefore, d can be ultimately interpreted as a measure of the degree of openness and maturity of an economy. We also consider the hypothesis of an economy facing negative institutional risk premium, $d < 0$. We consider that strong market sentiment may drive institutional risk premium to be negative, when certain institutional macroeconomic scenarios arising from international liquidity bias, strong domestic bias towards home assets and specific international institutional frameworks are fulfilled for a given economy. We provide a detailed discussion on this matter in the context of existence of investment adjustment costs and detail four possible dynamic setups with a relevant economic interpretation.

Agents take investment decisions on domestic assets and face convex investment adjustment costs on these decisions, given by function $\Omega(t)$, following the famous Hayashi [37] proposal:

$$\Omega(t) = i(t)\left(1 + \frac{h}{2}\frac{i(t)}{k(t)}\right). \tag{4}$$

In a closed economy framework convex investment adjustment costs are usually interpreted in the context of installation costs. In an open economy framework, the installation cost parameter, $h \in \Re$, has the following

[15] Stiglitz and Weiss [79] have shown that even in cases of individual borrowing, because of informational asymmetries or problems associated with moral hazard, risk premium or credit constraints, or both, are known to exist.

interpretation: if (i) $h < 0$, institutional conditions impose bias on investment in domestic assets, if (ii) $h > 0$, institutional conditions impose bias on investment in foreign assets. This mechanism is linked to the previous discussion on the degree of openness and maturity of an economy. Empirical evidence suggests the predominance of bias towards investment in domestic assets. This is known as the equity home bias puzzle. Evidence on this phenomena was first brought forward by French and Poterba [28]. Since mature developed economies offer smaller costs on investment in international assets, following our assumption on the higher degree of sophistication of its financial sector industry, we can assume that these economies face institutional conditions that promote smaller bias towards investment internationalization. The opposite is expected in less developed economies, where institutional conditions impose higher costs on investment in foreign assets. On the other hand, it is widely known that economies facing dire financial conditions, due to severe balance of payment imbalances leading to currency crises, increase the incentives for households to substitute domestic assets by foreign assets. Capital flights in this fashion are a consequence of domestic asset devaluation arising from currency value collapse and the consequent inflationary dynamics, which drive down the value of domestic assets against foreign assets. Although we don't consider currency in our model, we can consider that such extreme situations impose extraordinary institutional conditions, which lead to bias on investment in foreign assets.

Four dynamic scenarios with relevant economic interpretation can be considered, when we take into account the interactions between these two institutional parameters. As discussed in the previous paragraphs, the expected scenario, according to economic theory, is given by an economy facing positive risk premium and bias towards domestic assets, $d > 0$ and $h < 0$. However, there are empirical and theoretical grounds to assume that an economy may benefit from both negative risk premium and bias towards home assets, $d < 0$ and $h < 0$. We consider two institutional frameworks that may produce macroeconomic outcomes consistent with this specific scenario: (i) Flight to liquidity driven by reserve currency status; and (ii) Excess liquidity arising from international capital flows. The first scenario arises in economies with currencies that function as strategic reserve assets in international capital markets. Historically, this status has been held by the UK pound during the gold standard period and afterwards by the US dollar following the second world war and the Bretton Woods agreement. For historical, economic and geostrategic reasons, these two economies benefited from international financial bias, which resulted in higher demand and increased liquidity in both foreign exchange and sovereign bond markets. Longstaff [49] provides evidence that during liquidity flights arising from international financial crisis, investors are willing to pay a premium to hold US bonds. This strong liquidity effect may lead to negative risk premium scenarios. According to Ludvigson and Ng [53] sovereign bond markets are strongly driven by market sentiment, which leads to an acyclical behavior of risk premium. The authors give evidence of acyclical and negative risk premium dynamics in the US sovereign bond market and attribute this behaviour to investor decisions driven by market sentiments and macroeconomic factors. The authors link this outcome to theories that sustain that investors demand compensation for increased risks during economic downturns, which drives risk premium higher, and relax these demands during expansions, where risks are considered to be smaller. Other currencies have also benefited from reserve currency status and have been accepted by international investors as substitutes to the US dollar in recent decades. Besides the UK pound, we can include in this set of currencies the Deutsche Mark, now replaced by the Euro, the Japanese Yen and the Swiss Franc. The Japanese case is of special interest to our discussion, since it is linked to a strong home bias on domestic assets and a liquidity trap environment driven by historically low interest rates. Goyal and McKinnon [33] provides empirical evidence on Japan's consistent negative risk premium on sovereign bonds and links this outcome to the strong home bias on domestic assets effect mixed with the low interest rates environment in a context of an ineffective monetary policy. The common economic factors shared by the above mentioned economies are long run growth, export capacity, credit worthiness and creditor protection, strong property rights and historical low to moderate inflation. The second scenario arises in economies that benefit from strong international liquidity flows, which were driven by international low interest rates and resulted in a distortion of domestic bond markets, due to lower perceived default risk and improved creditworthiness. Agenor [1] provides an insightful theoretical discussion on this issue and maintains that this was the main cause driving the boom and bust of Asian economies during the nineties. The European periphery countries experienced the same environment with the introduction of the Euro and the period of low interest rates that followed. Again excess liquidity drove risk premium to low, and most likely negative levels, due to a perceived increased creditworthiness and lower default risk. This effect was a result of market perceptions about European institutional developments, which fuelled the belief that exchange rate risk between EU nations had vanished. During several years, European periphery countries yields on sovereign bonds were historically low and even negative, when compared to benchmark German sovereign bonds. This link was broken in the aftermath of the 2008 financial crisis, and since then economic factors have dominated international investors' decisions and bond market outcomes, leading to a return to positive spreads relative to German bonds. Finally, some open economies benefit from the status of commodity currencies, due to their strategic importance for world commodity markets. In recent decades the Canadian, Australian and New Zealand dollar, as well as the Norwegian Krone have benefited from this specific status. These economies usually experience excess liquidity driven by international financial flows during strong expansion periods. Rising demand for strategic commodities in world markets, leads to a rising demand on commodity currency assets, which results in currency appreciation and increased liquidity in domestic bond markets. Excess liquidity of foreign reserves may lead to severe distortions on risk premium in the absence of appropriate institutions. In recent decades, several commodity exporting economies followed the Norwegian institutional framework and constituted sovereign wealth funds, with the objective of investing commodity based revenues in foreign assets, to avoid distortions in domestic markets arising from excess foreign reserve liquidity.

We conclude this discussion on institutional scenarios with a description of macroeconomic nightmare scenarios arising in economies facing positive risk premium and bias towards foreign assets, $d, h > 0$. Recall that we discussed previously that scenarios consistent with $h > 0$, can be related to balance of payment crisis and expectations of currency crisis and debt default scenarios. In this critical environment, international investors price in this risk assuming strong probability of losses and demand a higher premium to hold the stressed economy sovereign bonds. Before the default scenario becomes inevitable, countries seek to lower the premium demanded for their bonds by guaranteeing debt roll over through bilateral agreements. This is usually arranged through IMF intervention and the implementation of structural adjustment programs. Loans are guaranteed by IMF stockholders and the soundness of the institutional arrangement is monitored by IMF economists. This institutional arrangement seeks to roll over debt repayments, until market risk premium on domestic bonds returns to affordable levels and there is no longer bias towards foreign assets. This institutional arrangement seeks to avoid macroeconomic nightmare scenarios, by guaranteeing a temporary debt subsidy at a negative real premium, for an economy facing bias towards foreign assets, $d < 0$ and $h > 0$. In recent years IMF interventions have come under severe criticism because of its consistent inability to achieve the desired goals and leaving economies worst off. Some authors suggest that institutionally imposed negative risk premiums scenarios creates moral hazard incentives for both the creditor and debtor. These authors argue that such arrangements promote the delay of economic adjustment by the debtor and reduce negotiation willingness of creditors. This noncooperative situation delays the achievement of a permanent solution to the unsustainable debt problem and usually results in higher costs for both debtors and creditors. Miller and Zhang [60] and Corsetti et al. [21] discuss this problem in detail and propose a standstill solution, or debt repayment freeze, during the economic adjustment program period, with the purpose of reducing the moral hazard consequences of negative imposed institutional risk premium. Since our model cannot predict such outcomes, it is of far more importance in this context to understand how international investors systematically fail to forecast unsustainable debt dynamics and price in the increasing risk on demanded premium, before the situation becomes irreversible. We believe that Ludvigson and Ng [53] hypothesis of strong market sentiment driving acyclical risk premium dynamics is a consistent explanation of this phenomenon. During good times investors fail to scrutinize correctly real risk premium and allow economies to accumulate excessive debt. The negative risk valuation of an economy debt dynamics functions as an incentive to continue to accumulate excessive debt, because it allows for short run economic and political gains. As soon as the situation deteriorates, investors penalize this behaviour and demand higher risk premium on the country bonds. Rising risk premium leads to devaluation and increased inflation expectations by domestic investors, who eventually bias their investments towards foreign assets. At this point IMF interventions provide temporary liquidity through debt subsidies and again guarantee negative risk premium, but now in an environment with bias towards foreign assets. Whatever the outcome of the adjustment program, sustainable long run growth dynamics are only achieved when bias towards domestic assets is restored. At this point investors will still be vigilant of a country's debt dynamics and demand a positive risk premium on bonds. Market sentiment now penalizes this economy. Eventually international investors' memory fades and this risk premium cycle can potentially restart. Although our proposal does not account for risk premium dynamics, we propose to study this phenomena assuming all these four scenarios separately.

We conclude the presentation of our economy with the definition of domestic and net foreign capital dynamics. Agents receive capital returns, $r_k \in \Re^+$, on domestic assets equal to the marginal productivity of firms, following the usual neoclassical assumption on market clearing conditions for perfectly competitive domestic capital markets. The marginal returns on domestic capital are given by the exogenous technology rate of firms,

$$r_k = \frac{\partial y(t)}{\partial k(t)} = A. \tag{5}$$

We can now write the intertemporal budget constraint for the representative agent, in terms of foreign debt/assets accumulation. This constraint is given by the following differential equation,

$$\dot{b}(t) = c(t) + i(t)\left(1 + \frac{h}{2}\frac{i(t)}{k(t)}\right) + rb(t)\left(1 + \frac{d}{2}\frac{b(t)}{k(t)}\right) - r_k k(t). \tag{6}$$

Firms accumulate capital following agents' investment decisions and face a depreciation rate of their capital stock equal to $\delta \in \Re^{+}$, following the usual linear differential specification for capital dynamics,

$$\dot{k}(t) = i(t) - \delta k(t). \tag{7}$$

In our setup, we assume that agents can have temporary disinvestment decisions, in order to improve their foreign net assets balances or increase their consumption levels. In the long run, we assume that the following asymptotic condition is fulfilled:

$$\liminf_{t\to\infty} \frac{i(t)}{k(t)} > \delta. \tag{8}$$

We finish the presentation of this economy with the description of the dynamic optimization problem faced by the aggregate economy, which is identical to the solution of the aggregate representative agent and central planner dynamic optimization problems. Recall that aggregation in a representative agent framework is given by assuming $X_i(t) = Nx_i(t)$, where $i \in \{1, ..., 4\}$, and $X_i(t)$, $x_i(t)$ correspond to each of our aggregate and individual variables, consumption, $c(t)$, net foreign assets, $b(t)$, domestic capital, $k(t)$, and investment, $i(t)$, respectively. The aggregate solution is given by the maximization of the aggregate objective function given in (9), subject to the aggregate state conditions given by (6) and (7), which do not depend on N. Although the optimality condition for aggregate consumption still depends on the size of the economy, when we solve the representative agent or central planner problem, this parameter disappears when we define the *Keynes-Ramsey* consumption equations. Therefore, we can simplify the dynamic optimization problem of this economy by assuming the following aggregate representation[16],

$$\begin{aligned}
&\underset{C(t),I(t)}{MAX} \int_0^{\infty} e^{-\rho t} C(t)^{\gamma}\, dt \\
&\text{subject to the solution of:} \\
&\dot{B}(t) = C(t) + I(t)\left(1 + \tfrac{h}{2}\tfrac{I(t)}{K(t)}\right) + rB(t)\left(1 + \tfrac{d}{2}\tfrac{B(t)}{K(t)}\right) - r_k K(t); \\
&\dot{K}(t) = I(t) - \delta K(t);
\end{aligned} \tag{9}$$

satisfying the transversality conditions (A.8) and (A.9) of Appendix A, guaranteeing that solutions to (9) do not grow too fast.

3. Stationary dynamics for the aggregate economy

To derive the relevant dynamical system describing the optimal solution to (9), we first derive in Appendix A the *Pontryagin* necessary first order conditions, which are given in (A.2) to (A.7). These conditions are sufficient if they fulfil admissibility conditions, given in (A.10), and transversality conditions, (A.8) and (A.9), following the seminal result by Arrow and Kurtz [3]. We start the derivation of this optimal control problem by taking the time derivatives of the optimality conditions, given by (A.2) and (A.3). We obtain the following expressions:

$$\dot{\lambda}(t) = -\gamma(\gamma - 1) C(t)^{\gamma-2} \dot{C}(t); \tag{10}$$

$$\dot{q}(t) = -\dot{\lambda}(t)\left(1 + hI(t)K(t)^{-1}\right) - \lambda(t) hK(t)^{-1} \dot{I}(t) + \lambda(t) hI(t) K(t)^{-2} \dot{K}(t). \tag{11}$$

We then substitute these expressions and the optimality conditions, in the co-state conditions (A.6) and (A.7), and obtain the two possible *Keynes-Ramsey* optimal consumption rules for this economy[17], $\dot{C}_B(t)$ and $\dot{C}_K(t)$[18],

[16] This dynamic problem yields the same solution as the central planner and representative agent problems and, therefore, represents a slight simplification of the economic growth problem described in the previous paragraphs. This result can be easily confirmed following our definitions in Appendix A. We discard the demonstration of this result in order to contain our already long presentation.

[17] By *Keynes-Ramsey* consumption rules, we mean the intertemporal dynamic consumption decisions that are obtained for this control variable in an optimal control problem with a constant intertemporal discount rate. In macroeconomics literature these dynamic equations are known by *Keynes-Ramsey* consumption rules, following the work by the two famous Cambridge scholars, that related intertemporal consumption decisions with the discounted value of expected future incomes and optimal savings for capital accumulation. It is our opinion that in open economy optimization problems with two state variables, this rule is not unique, since state defined income accumulation can vary in its source. Therefore it is reasonable to impose two possible consumption paths that satisfy the optimal investment condition. In this model, optimal investment decisions impose an indifference rule on the intertemporal marginal adjustment between different assets, in order to allow for distinct capital accumulation decisions. This mechanism has the following interpretation, investors will always choose to accumulate assets that adjust faster to optimum outcomes rather than invest in assets that yield longer adjustment rates. In economics jargon the co-state variables represent the shadow price (or marginal value) of a specific asset.

[18] To obtain the second *Keynes-Ramsey* consumption rule, $\dot{C}_K(t)$, it is convenient to start by substituting the optimality condition for consumption, (A.2), and its time derivative, (10), in the optimality condition for investment, (A.3), and in its time derivative, (11).

arising from consumption driven either by net foreign assets accumulation or domestic capital accumulation,

$$\dot{C}_B(t) = \frac{C(t)}{\gamma - 1}\left(\rho - r - rd\frac{B(t)}{K(t)}\right), \tag{12}$$

$$\dot{C}_K(t) = \frac{C(t)}{(\gamma-1)\left(1+\frac{hI(t)}{K(t)}\right)}\left[-\frac{h\dot{I}(t)}{K(t)} + \frac{hI(t)\dot{K}(t)}{K(t)^2} + \left(1+\frac{hI(t)}{K(t)}\right)(\rho+\delta) - \frac{rdB(t)^2}{2K(t)^2} - \frac{hI(t)^2}{2K(t)^2} - r_k\right]. \tag{13}$$

Optimal investment decisions in this economy are given by imposing $\dot{C}_B(t) = \dot{C}_K(t)$. After some fair amount of calculus and the substitution of the state condition for capital accumulation, (A.7), we obtain the differential equation driving investment activities[19],

$$\dot{I}(t) = \frac{I(t)^2}{2K(t)} + \left(r + rd\frac{B(t)}{K(t)}\right)I(t) - \frac{rd}{2h}\frac{B(t)^2}{K(t)} + \left(r + rd\frac{B(t)}{K(t)} + \delta - r_k\right)\frac{K(t)}{h}. \tag{14}$$

This economy is thus defined by the dynamical system given by the differential equations of the controls in consumption and investment, (12) and (14), and the state conditions for net foreign financial assets and domestic capital accumulation, defined by (A.6) and (A.7). We define a stationary dynamical system by taking advantage of the scaled invariance of the dynamics, and redefine the variables, $X_i(t)$, in terms of domestic capital units:

$$Z_i(t) = \frac{X_i(t)}{K(t)} \Longrightarrow \dot{Z}_i(t) = \frac{\dot{X}_i(t)}{K(t)} - \frac{X_i(t)}{K(t)}\frac{\dot{K}(t)}{K(t)}, \tag{15}$$

where i reduces to $i \in \{1,2,4\}$ and $Z_i(t)$ defines scaled consumption, net foreign assets and investment, respectively. Following this rule, we redefine the system in terms of the scaled controls and scaled state equations:

$$\dot{Z}_1(t) = Z_1(t)\left(\frac{\rho - r(1 + dZ_2(t)) + (\delta - Z_4(t))(\gamma - 1)}{\gamma - 1}\right); \tag{16}$$

$$\dot{Z}_2(t) = Z_1(t) + Z_4(t)\left(1 + \frac{h}{2}Z_4(t)\right) + \left(r + \frac{rd}{2}Z_2(t) + \delta - Z_4(t)\right)Z_2(t) - r_k; \tag{17}$$

$$\dot{Z}_4(t) = -\frac{1}{2}Z_4(t)^2 + (r + rdZ_2(t) + \delta)Z_4(t) - \frac{rd}{2h}Z_2(t)^2 + (\delta + r + rdZ_2(t) - r_k)\frac{1}{h}. \tag{18}$$

Since the system is now independent of domestic capital dynamics, which only depends endogenously on investment outcomes, we have reduced the dynamics of this economy to three dimensions. Domestic capital is given by the following expression,

$$K(t) = K(0)\,e^{\int_0^t (Z_4(s) - \delta)ds}, \tag{19}$$

following the result in (8). We now introduce the notion of acceptable solutions to the system given by (16), (17) and (18). We assume that such solutions can be described as an ergodic invariant set[20] with a well defined invariant probability measure, whose expectation operator we denote by $\langle\rangle$, such that $\lim_{t\to\infty} t^{-1}\int_0^t Z_i(t) \to \langle Z_i\rangle$. Following this definition, we can redefine domestic capital dynamics in the long run by taking the asymptotic limit of expression (19). We obtain

$$\lim_{t\to\infty} t^{-1}\log K(t) = \langle Z_4\rangle - \delta. \tag{20}$$

Following our definition of solutions given by invariant sets, we can now define the constraint on scaled investment activities that imposes the existence of long run growth dynamics, $\lim_{t\to\infty} K(t) = \infty$, exponentially, $\langle Z_4\rangle > \delta$. In order to guarantee the existence of an optimum solution arising from the *Pontryagin* maximum conditions, we need to check under which circumstances the transversality conditions are fulfilled. For that purpose we rearrange expressions (A.8) and (A.9) in terms of scaled variables and substitute the co-state variables from the optimality conditions (A.2) and (A.3). The transversality conditions are now given by:

$$\lim_{t\to\infty} -\gamma(Z_1(t)K(t))^{\gamma-1}Z_2(t)K(t)e^{-\rho t} = 0; \tag{21}$$

[19] We would like to stress that this result is independent of our interpretation of indifference between optimal consumption strategies. The same condition defining investment dynamics is obtained when substituting directly (12) while deriving (14).

[20] By invariant set we refer to solutions of the scaled dynamical system that can be defined as distributions obtained from bounded trajectories of $\lim_{t\to\infty} Z_i(t)$ in the phase plane. We follow the definition of an invariant set composed by asymptotic limit sets of points given in Guckenheimer and Holmes [36]. Let ϕ_t be a flow such that the α limit set of x for ϕ_t is the set of accumulation points of $\phi_t(x), t \to -\infty$. The ω limit set of x for ϕ_t is the set of accumulation points of $\phi_t(x), t \to \infty$. The α and ω limits of x are its asymptotic limit sets.

$$\lim_{t\to\infty} \gamma\left(Z_1(t)K(t)\right)^{\gamma-1}\left(1+hZ_4(t)\right)K(t)e^{-\rho t}=0. \tag{22}$$

Recall that we defined domestic capital dynamics, $K(t)$ in (19), as a function of scaled investment dynamics, $Z_4(t)$. Assuming that we only accept solutions for the dynamical system defined in (16) to (18), given by invariant sets, following the result in (20), we can rearrange the transversality conditions given in (21) and (22) in a intuitive fashion by taking the scaled limit of the logarithm of (21) and (22), and solving the transversality constraints as an asymptotic inequality[21]. The transversality conditions are now given by:

$$\lim_{t\to\infty} t^{-1}\log\left[-\gamma\langle Z_1\rangle^{\gamma-1}K(0)^{\gamma}\langle Z_2\rangle e^{\left[(\gamma-1)(\langle Z_4\rangle-\delta)+\langle Z_4\rangle-\delta-\rho\right]t}\right]<0; \tag{23}$$

$$\lim_{t\to\infty} t^{-1}\log\left[\gamma\langle Z_1\rangle^{\gamma-1}K(0)^{\gamma}\left(1+h\langle Z_4\rangle\right)e^{\left[(\gamma-1)(\langle Z_4\rangle-\delta)+\langle Z_4\rangle-\delta-\rho\right]t}\right]<0. \tag{24}$$

From (23) or (24) it is straightforward to obtain the transversality constraint for the existence of an optimal solution as a function of the invariant probability measure describing scaled investment trajectories. Given the long run growth restriction, (20), the optimal growth constraint for the problem defined in (9) is the interval,

$$\delta<\langle Z_4\rangle<\delta+\frac{\rho}{\gamma}. \tag{25}$$

4. Steady states, linearized dynamics and local stability conditions

We now turn our attention to the study of steady states and local qualitative dynamics. The dynamical system described by (16), (17) and (18) has two sets of steady states with specific economic meaning. We define the complete set of steady states as $\widetilde{\widetilde{Z}}=\left\{\widetilde{Z}^{*},\widetilde{Z}^{**}\right\}$, where the first set of steady states, $\widetilde{Z}^{*}$, is obtained by setting $Z_1^{*}=0$, and the second set of steady states, $\widetilde{Z}^{**}$, is obtained assuming $Z_1^{**}\neq 0$.

The first set of steady states is given by the intersection of two quadratic curves defined by the system, $\dot{Z}_2(t)\wedge\dot{Z}_4(t)=0$. The derivation of $\widetilde{Z}^{*}$ appears to require the solution of a fourth order equation. It can be solved using a numerical polynomial solver routine. In Appendix B, we provide the detailed description of an efficient and accurate algorithm to perform this computation. Alternatively, one can note that the two quadratics, $\left(Z_2^{*},Z_4^{*}\right)$ happen to have the same center, $\left(Z_{2,0}^{*},Z_{4,0}^{*}\right)=\left(1+h(rd+\delta+r)(1-hrd)^{-1},(rd+\delta+r)(1-hrd)^{-1}\right)$. Assuming the transformation $z_2^{*}=Z_2^{*}-Z_{2,0}^{*}$ and $z_4^{*}=Z_4^{*}-Z_{4,0}^{*}$, the system, $\dot{Z}_2(t)\wedge\dot{Z}_4(t)=0$ reduces to:

$$\frac{rd}{2}\left(z_2^{*}\right)^2+\frac{h}{2}\left(z_4^{*}\right)^2-z_2^{*}z_4^{*}-C_2=0; \tag{26}$$

$$-\frac{rd}{2h}\left(z_2^{*}\right)^2-\frac{\left(z_4^{*}\right)^2}{2}+rdz_2^{*}z_4^{*}-C_4=0; \tag{27}$$

where C_2 and C_4 are given by the following parameter expressions:

$$C_2=Z_{4,0}^{*}\left(1+\frac{h}{2}Z_{4,0}^{*}\right)+\left(r+\frac{rd}{2}Z_{2,0}^{*}+\delta-Z_{4,0}^{*}\right)-r_k; \tag{28}$$

$$C_4=Z_{4,0}^{*}\left(r+rdZ_{2,0}^{*}+\delta-\frac{Z_{4,0}^{*}}{2}\right)-\frac{rd}{2h}Z_{2,0}^{*}+\frac{r+rdZ_{2,0}^{*}+\delta-rk}{h}. \tag{29}$$

Multiplying (27) by h, the solution to z_2^{*} in terms of is z_4^{*} obtained after adding expressions (26) and (27),

$$z_2^{*}=\frac{-\left(C_2+hC_4\right)}{1-rdh}\frac{1}{z_4^{*}}. \tag{30}$$

Substituting back (30) in (27), solutions to z_4^{*} are given by the resulting biquadratic equation

$$\frac{(rdh-1)^2}{2}\left(z_4^{*}\right)^4+\left(rdC_2+C_4\right)(rdh-1)\left(z_4^{*}\right)^2+\frac{rd}{2h}\left(C_2+hC_4\right)=0, \tag{31}$$

[21] Recall that a dynamic process that scales exponentially, $w(t)\sim e^{\psi t}$, can be defined asymptotically in the following fashion, $\lim_{t\to\infty}t^{-1}\log w(t)=\psi$. If $\psi>0\Rightarrow w(t)\to\infty$. If $\psi<0\Rightarrow w(t)\to 0$.

whence,

$$z_4^* = \pm \frac{\sqrt{-\left(rdC_2 + C_4\right) \pm \sqrt{rdh^{-1}\left(rdh - 1\right) C_2^2 + \left(1 - rdh\right) C_4^2}}}{rdh - 1} \tag{32}$$

In the case of the economic feasible steady states, $\widetilde{Z}^{**}$, the solution can be obtained analytically by solving the quadratic equation given by $\dot{Z}_4\left(t\right) = 0$, after substituting by the solution of $\dot{Z}_1\left(t\right) = 0 \wedge Z_1^{**} \neq 0$. This operation yields the following quadratic equation for Z_4^{**},

$$\left[-\tfrac{(\gamma-1)^2}{2hrd} - \gamma + \tfrac{1}{2}\right]\left(Z_4^{**}\right)^2 + \left[\rho + \delta\gamma + \tfrac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd} - \tfrac{\gamma-1}{h}\right] Z_4^{**} + \rho + \delta\gamma - \tfrac{(\rho-r+\delta(\gamma-1))^2}{2rd} - r_k = 0. \tag{33}$$

The solution to (33), defining the economic feasible steady state for Z_4^{**} is thus given by

$$Z_4^{**} = \frac{-\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right) \pm \sqrt{\left(\rho+\delta\gamma+\frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd}-\frac{\gamma-1}{h}\right)^2 - 4\left[\left(-\frac{(\gamma-1)^2}{2hrd}-\gamma+\frac{1}{2}\right)\left(-\frac{(\rho-r+\delta(\gamma-1))^2}{2rd}+\rho+\delta\gamma-r_k\right)\frac{1}{h}\right]}}{-\frac{(\gamma-1)^2}{hrd}-2\gamma+1}. \tag{34}$$

Scaled consumption and net financial asset equilibrium expressions can be computed in terms of Z_4^{**} solutions. These conditions are given in (35) and (36), below:

$$Z_2^{**} = \frac{\rho - r + \delta\left(\gamma - 1\right)}{rd} - Z_4^{**}\frac{\left(\gamma - 1\right)}{rd}; \tag{35}$$

$$Z_1^{**} = r_k - Z_4^{**}\left(1 + \frac{h}{2} Z_4^{**}\right) - \left(r + rdZ_2^{**} + \delta - Z_4^{**}\right) Z_2^{**}. \tag{36}$$

We continue the analytical discussion of our dynamical system with the definition of general *Jacobian* matrix for this system, evaluated in the vicinity of a given fixed point, $\bar{Z}_i \in \widetilde{Z}$. The general *Jacobian* is defined by

$$J = \begin{bmatrix} \frac{\rho - r\left(1+d\bar{Z}_2\right)+\left(\delta-\bar{Z}_4\right)(\gamma-1)}{\gamma-1} & -\frac{rd}{\gamma-1}\bar{Z}_1 & -\bar{Z}_1 \\ 1 & r + rd\bar{Z}_2 + \delta - \bar{Z}_4 & 1 + h\bar{Z}_4 - \bar{Z}_2 \\ 0 & rd\bar{Z}_4 - \frac{rd}{h}\left(\bar{Z}_2 - 1\right) & -\bar{Z}_4 + r + rd\bar{Z}_2 + \delta \end{bmatrix}. \tag{37}$$

The generalized characteristic equation, $\det\left(J - \Lambda I\right) = 0$, for this *Jacobian* matrix comes to

$$\left(J_{1,1} - \Lambda\right)\left(J_{2,2} - \Lambda\right)\left(J_{3,3} - \Lambda\right) + J_{1,3}J_{3,2} - J_{3,2}J_{2,3}\left(J_{1,1} - \Lambda\right) - \left(J_{3,3} - \Lambda\right) J_{1,2} = 0, \tag{38}$$

where Λ is the eigenvalue solution to (38) and I the identity matrix. The following simplifications can be assumed for each set of fixed points: (i) $J_{1,2}, J_{1,3} = 0$, $J_{2,2} = J_{3,3}$ and $J_{3,2} = J_{2,3}rd/h$, for the set of fixed points given by $\widetilde{Z}_i^*$; and (ii) $J_{1,1} = 0$, $J_{2,2} = J_{3,3}$, $J_{3,2} = J_{2,3}rd/h$ and $J_{1,2} = J_{1,3}rd/\left(\gamma - 1\right)$, for the set of fixed points given by $\widetilde{Z}_i^{**}$. In Appendix C we provide a general description of linearized dynamics and define hyperbolicity conditions in the vicinity of each specific set of steady states for this system.

To put forward sufficient conditions guaranteeing local stability of economic feasible solutions, $\widetilde{Z}_i^{**}$, we resort to the *Routh-Hurwitz Criterion*, following the seminal paper by Hurwitz [41]. The *Routh-Hurwitz Criterion* guarantees that all solutions to a polynomial of degree n have a negative real part. The advantage of following this approach is that it allows us to impose local stability conditions without having to compute the eigenvalues of (C.1). To determine the signs of the solutions of a cubic polynomial, we start by defining generically (C.2) as,

$$a_0\left(\Lambda^{**}\right)^3 + a_1\left(\Lambda^{**}\right)^2 + a_2\Lambda^{**} + a_3 = 0. \tag{39}$$

The *Hurwitz* matrices for a cubic polynomial are generically given by:

$$H_0 = \left[a_0\right], H_1 = \left[a_1\right], H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix}, H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}. \tag{40}$$

The *Routh-Hurwitz Criterion* guarantees that solutions of the polynomial defined in (39) have a negative real part, $Re\left(\Lambda^{**}\right) < 0$, if the determinants of the *Hurwitz* matrices are positive, $\left|H_0\right|, \left|H_1\right|, \left|H_2\right|, \left|H_3\right| > 0$. Given this definition, sufficient conditions for local stability are given by:

$$\begin{gathered} \left|H_0\right| = a_0 = 1 > 0 \\ \left|H_1\right| = a_1 = -2J_{2,2}^{**} > 0 \Longrightarrow J_{2,2}^{**} < 0 \\ \left|H_2\right| = a_1a_2 - a_3 = 2J_{2,2}^{**}\left(\tfrac{rd}{\gamma-1}J_{1,3}^{**} - \left(J_{2,2}^{**}\right)^2 + \tfrac{rd}{h}\left(J_{2,3}^{**}\right)^2\right) + rdJ_{1,3}^{**}\left(\tfrac{1}{h}J_{2,3}^{**} - \tfrac{1}{\gamma-1}J_{2,2}^{**}\right) > 0 \\ \left|H_3\right| = a_3\left|H_2\right| > 0 \Longrightarrow a_3 = -rdJ_{1,3}^{**}\left(\tfrac{1}{h}J_{2,3}^{**} - \tfrac{1}{\gamma-1}J_{2,2}^{**}\right) > 0 \end{gathered} . \tag{41}$$

We now focus on the evaluation of local qualitative dynamics in the context of a broad parameter space. We start with the definition of an algorithm for the exploration of a parameter space defined by a vector of parameters, $\widetilde{\mu}$, for a given parameter space with j parameters, with individual parameter sets defined as, $\mu_j = \left[\mu_j^{min} \;\; \mu_j^{max}\right]$. A robust algorithm that maps all possible outcomes, is given by a grid search of the complete parameter space, such that the grid search has a small enough step size. According to the linearization theorem, qualitative dynamics are robust to small parameter changes in hyperbolic autonomous dynamical systems. Following this definition, the individual parameter set can be defined as the sum of all its partitions, $\sum_{m=1}^{p} \mu_{m,j}$, where m is the index of each equal partition, $\mu_{m,j}$, of the original parameter set, μ_j. Assuming that we choose a large enough number of partitions, p, such that each partition of the parameter space is small enough and therefore robust under the linearization theorem, we can define the following general measure for the total size of the parameter space as, $\prod_j \sum_m \mu_{m,j}$. Now, for example, if we consider that a grid search with a step size equal to 10^{-2} is consistent with the previous definition, the total parameter space to explore assuming $\rho, \gamma, \delta, r, r_k \in [0\;1]$ and $d, h \in [-10\;10]$, requires a grid search procedure that performs $4 \cdot 10^{16}$ iterations, in order to cover all possible parameter combinations. Given that this is not a feasible computational task, we propose to explore this vast parameter space assuming a stochastic variation of the grid search procedure described. Instead of grid searching each possible combination, we propose to draw parameter combinations stochastically, assuming a uniform distribution of the parameter space, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$ and $d, h \sim U(-10,10)$. If we draw large enough samples of uniformly random distributed numbers for a given parameter space, then the total parameter space covered by the samples will asymptotically approach the original parameter space. Therefore, we can define an accurate probability measure of a given event, by computing the sample averages of parameter combinations consistent with these events.

The first conclusion drawn from the application of the stochastic search routine, for the parameter space defined in the previous paragraph, is the absence of local stable solutions. Several samples[22] of size 10^9 were computed and not a single outcome satisfied both the optimal growth condition and the *Routh-Hurwitz criterion*[23]. The qualitative dynamics of steady-states consistent with (25) can be characterized[24] in the following fashion: (i) if $d > 0$, there are only saddle solutions with stable dimension equal to one and these correspond to the positive root of (34); and (ii) if $d < 0$, we can have saddle solutions with stable dimension equal to two for the positive root of (34). This last outcome is more likely to occur when there is a small bias towards home assets. Further, when $d < 0$, there are parameter combinations where the negative root of (34) is a saddle solution of stable dimension equal to one consistent with (25). These two saddle solutions may coexist for specific institutional scenarios. We discuss this result further in section 7. In the next sections, we provide several examples of the application of this stochastic method and, when convenient, portray some of the sample results obtained.

5. Local singularities: Saddle-node, Hopf and Fold-Hopf bifurcations

We now turn our attention to the description of bifurcations arising from the set of economic feasible fixed points. We start by describing the conditions required for the existence of saddle-node bifurcations also known as folds. A saddle-node bifurcation is a co-dimension one singularity that imposes dramatic qualitative changes in the system behaviour. It occurs when two fixed points collide and disappear. This bifurcation is associated with dramatic dynamic phenomena, such as hysteresis or catastrophe. In the vicinity of this bifurcation, small parameter perturbations may provoke changes in the phase-space organization and give rise to path dependence and global nonlinear phenomena, such as heteroclinic and homoclinic orbits. A recent example of this bifurcation in a continuous time model of endogenous growth is found in Chen and Guo [19]. Saddle-node bifurcations arise in regions where an equilibrium is at a branching point, one of the eigenvalues is equal to zero and the remaining eigenvalues are real. Following our steady state formulae for economic feasible outcomes, given in (34) to (36), an optimal candidate for a saddle-node bifurcation is the parameter constraint that guarantees the square root term in (34) is equal to zero. For convenience, we choose to vary r_k. At the branching point of Z_4^{**}, the bifurcation parameter, r_k^{**}, is equal to,

$$r_k^{**} = -\frac{h}{4}\left[\frac{\left(\rho + \delta\gamma + \frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd} - \frac{\gamma-1}{h}\right)^2}{-\frac{(\gamma-1)^2}{2hrd} - \gamma + \frac{1}{2}}\right] + \left[\rho + \delta\gamma - \frac{(\rho - r + \delta(\gamma-1))^2}{2rd}\right]. \tag{42}$$

[22]We compute samples using a C routine compiled with the GSL scientific library.

[23]This result is confirmed by numerical computation of eigenvalues.

[24]To determine the qualitative dynamics of the fixed points we assumed that: (i) saddles with stable dimension equal to one are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) < 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) < 0$, (ii) saddles with stable dimension equal to two are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) > 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) < 0$, and (iii) divergent solutions are consistent with $\prod_{i=1}^{3} Re\left(\Lambda_i^{**}\right) > 0$ and $\max\left(Re\left(\Lambda_i^{**}\right)\right) \cdot \min\left(Re\left(\Lambda_i^{**}\right)\right) > 0$.

Recall that equilibrium for Z_4^{**} is now given by:

$$Z_4^{**}\left(r_k^{**}\right) = \frac{-\left(\rho + \delta\gamma + \frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd} - \frac{\gamma-1}{h}\right)}{-\frac{(\gamma-1)^2}{hrd} - 2\gamma + 1}. \tag{43}$$

In Appendix D, we describe sufficient conditions for the existence of a saddle-node bifurcation. First, we recall a necessary condition that has to be fulfilled at the critical equilibrium point, $\det(J^{**}) = 0 \Rightarrow \Lambda_1^{**} = 0$. This condition is described in (D.1) and (D.2). In (D.3), we confirm that this condition is met at the branching point, defined in (43), which confirms the result in (42). To guarantee that the remaining condition for the existence of a saddle-node bifurcation is fulfilled, we solve the characteristic polynomial, (D.4), in the vicinity of (42), and obtain the remaining eigenvalues, Λ_2^{**} and Λ_3^{**}, defined in (D.5). This condition is given in (D.6). Substituting (D.6) with the *Jacobian* terms, and rearranging, the existence of a saddle-node bifurcation requires that: (i) when $d > 0$ we have $Z_1^{**}\left(r_k^{**}\right) > h\left[r + rdZ_2^{**}\left(r_k^{**}\right) + \delta - Z_4^{**}\left(r_k^{**}\right)\right]^2$; and (ii) when $d < 0$ we have $0 < Z_1^{**}\left(r_k^{**}\right) < h\left(r + rdZ_2^{**}\left[r_k^{**}\right) + \delta - Z_4^{**}\left(r_k^{**}\right)\right]^2$ and $h > 0$.

To confirm the existence of saddle-node bifurcations for this economy, we performed a numerical evaluation of possible outcomes, assuming that growth and optimality conditions are fulfilled for solutions in the feasible economic space. We computed samples following the stochastic sampling method discussed in the previous section[25]. The outcomes obtained suggest that saddle-node bifurcations are a common outcome for a broad range of parameter combinations and are more likely to occur in institutional scenarios where there is bias toward home assets, $h < 0$, and positive risk premium, $d > 0$. Numerical results suggest that when $d > 0$, we have $h > 0$ and $\Lambda_2^{**}\Lambda_3^{**} < 0$. When $d < 0$, numerical results suggest that $\Lambda_2^{**}, \Lambda_3^{**} > 0$.

We continue the discussion on local bifurcations with a description of analytical conditions for the existence of general and attracting *Hopf* bifurcations. The attracting *Hopf* bifurcation is usually related to the existence of limit cycles that can be observed physically. The existence of *Hopf* bifurcations in models of endogenous growth implies the coexistence of optimal growth and cycles. This literature has established itself in growth theory during the last two decades, in what has been established as optimal growth and cycle models. Some recent papers on this subject that follow the same base modeling assumptions of our proposal are the proposals by Slobodyan [77], Nishimura and Shigoka [63] and Wirl [89]. Examples of earlier literature on this subject can be found in the papers by Lordon [50], Greiner and Semmler [35], Greiner [34], Drugeon [26], Benhabib and Nishimura [10] and Asada et al. [4].

General *Hopf* bifurcations require that the following set of eigenvalue conditions is fulfilled: (i) $\mathrm{Re}\left(\Lambda_{2,3}^{**}\right) = 0$; (ii) $\mathrm{Im}\left(\Lambda_{2,3}^{**}\right) \neq 0$. The existence of an attracting *Hopf* bifurcation requires additionally that $\mathrm{Tr}(J^{**}) < 0 \Rightarrow \mathrm{Re}\left(\Lambda_1^{**}\right) < 0$. For reasons of convenience, we define these conditions using the *Hurwitz Determinants* described in section 4. Following Liu [48], an attracting *Hopf* bifurcation for vector fields in $\mathfrak{R}^3$, occurs if the following generic conditions are fulfilled: (i) $|H_0(\mu^{**})|, |H_1(\mu^{**})| > 0$; (ii) $|H_2(\mu^{**})| = 0$; (iii) $a_3 > 0$; and (iv) $\partial|H_2(\mu^{**})|/\partial\mu \neq 0$. Where, μ is the bifurcation parameter and μ^{**} is the bifurcation parameter at the *Hopf* equilibrium point, which is obtained from the solution of the second condition. The last condition guarantees that the eigenvalues cross the imaginary axis with non-zero speed. A general *Hopf* bifurcation does not require that $|H_1(\mu^{**})| > 0$ is fulfilled. Given this set of general conditions, the existence of attracting *Hopf* bifurcations in our economy, requires that the following conditions are fulfilled:

$$\begin{gathered} -rdJ_{1,3}^{**}\left(\frac{1}{h}J_{2,3}^{**} - \frac{1}{\gamma-1}J_{2,2}^{**}\right) > 0; \\ 2J_{2,2}^{**}\left(\frac{rd}{\gamma-1}J_{1,3}^{**} - \left(J_{2,2}^{**}\right)^2 + \frac{rd}{h}\left(J_{2,3}^{**}\right)^2\right) + rdJ_{1,3}^{**}\left(\frac{1}{h}J_{2,3}^{**} - \frac{1}{\gamma-1}J_{2,2}^{**}\right) = 0; \\ J_{2,2}^{**} < 0. \end{gathered} \tag{44}$$

To test the coexistence of optimal growth and cycles, we explore numerically solutions consistent with $Z_1^{**} > 0$, (25) and (44), assuming r_k as bifurcation parameter. For this purpose, we adapted our routine, to explore a bifurcation interval, $0 < r_k < 1$, for a given stochastic combination of parameters[26]. As expected, the modified

[25]The results were obtained from a sample with 10^9 random draws of uniform distributed numbers, $\rho, \gamma, \delta, r \sim U(0,1)$ and $d, h \sim U(-10, 10)$. We restricted the outcomes of this sample to solutions where $0 < r_k^{**} < 1$.

[26]We explore supercritical *Hopf* bifurcations by varying parameter r_k along the interval $0 < r_k < 1$, assuming a step size iteration of magnitude 10^{-2}. The outcomes discussed in this section were obtained from a sample with 10^7 random draws, assuming that the remaining parameters are uniformly distributed as in previous experiments, $\rho, \gamma, \delta, r, \sim U(0,1)$ and $d, h \sim U(-10, 10)$. In this experiment we consider that the supercritical *Hopf* bifurcation point is given by the average value of the crossing interval, when there is a change in sign consistent with (44). Steady state outcomes take into account this adjustment.

stochastic search routine was not able to detect the existence of parameter combinations consistent with optimal growth dynamics undergoing attracting *Hopf* bifurcations[27]. Given this outcome, we focused our efforts on the detection of general *Hopf* bifurcations. The samples obtained show that only the positive root of Z_4^{**} is consistent with the existence of general *Hopf* bifurcations. This bifurcation scenario is more likely to occur when $d, h < 0$, but may also occur when $d < 0$ and $h > 0$. Finally, our results show that $0 < \Lambda_1^{**} < 1$ and in most of the cases small.

We finish this section with the description of necessary conditions required for the existence of a codimension two fold-hopf bifurcation in this system[28]. This bifurcation is born from the merging of the two previously discussed instabilities. When the saddle-node and *Hopf* bifurcation curves are tangential in the parameter space a fold-hopf bifurcation is born. This singularity is characterized in vector fields in $\mathfrak{R}^3$ by: (i) $\det(J^{**}) = 0 \Rightarrow \Lambda_1^{**} = 0$; (ii) $\mathrm{Tr}(J^{**}) = 0 \Rightarrow \mathrm{Re}\left(\Lambda_{2,3}^{**}\right) = 0$; and (ii) $\mathrm{Im}\left(\Lambda_{2,3}^{**}\right) \neq 0$. The presence of this bifurcation shows that there is a path towards complex dynamics in this system. Fold-Hopf points are associated with several nonlinear phenomena. The influence of this bifurcation is not limited to parameter regimes in its close vicinity, it stretches far beyond the specific bifurcation point and may give rise to a cascade of complex dynamic transitions, including the local birth of chaos. In this framework, policy analysis has to take into account the increased complexity of possible model outcomes.

Although the unfolding of a fold-hopf bifurcation scenario is not fully known and in some sense impossible to describe in all detail, four transition scenarios can be considered for flows in $\mathfrak{R}^3$. The first two scenarios imply subcritical transitions and no torus formation. The unfolding of the most simple of these scenarios may not be associated with global nonlinear phenomena, but at least one limit cycle is known to exist[29]. The remaining scenarios may arise with subcritical and supercritical transitions and give rise to torus bifurcations and complex global dynamics. These transitions may create saddle node bifurcations of periodic orbits on the invariant torus, torus breakdown and chaos, heteroclinic orbits on a sphere (heteroclinic cycles), bursting and Sil'nikov bifurcations leading to chaos. A formal definition of the exact unfolding scenarios for this system requires the computation of the normal form coefficients using numerical continuation software.

This bifurcation scenario has been gaining a greater attention in other fields of applied mathematics[30]. This has not been the case in the field of economic dynamics. We were only able to find one article where this topic is discussed in economic literature. Brito [12] proves the existence of fold-hopf bifurcations generally for optimal control problems with one control and three state variables, that have solutions given by flows in $\mathfrak{R}^3$.

We finish this discussion with a description of the sufficient conditions for the existence of the fold-hopf bifurcation. The easiest path to obtain these conditions is to continue equilibrium from the saddle-node bifurcation point. A necessary condition for the existence of a fold-hopf bifurcation is given by setting $J_{2,2}^{**}\left(r_k^{**}\right), J_{2,3}^{**}\left(r_k^{**}\right) = 0$, following the saddle-node condition (D.2). We have a fold-hopf bifurcation, given in terms of parameters r_k^{**} and ρ^{**}, when condition (D.8) is fulfilled, and provided that there is negative risk premium, $d < 0$, following the eigenvalue solution in (D.7). If $d > 0$ we have a neutral saddle with $\Lambda_1^{**} = 0$ and $\sum \Lambda_{2,3}^{**} = 0$. The second parameter constraint, ρ^{**}, is given in (D.9). Again we resort to our stochastic routine to map the the parameter space for this bifurcation[31]. Below in Figure 2 we portray the sampling outcomes obtained for solutions consistent with fold-hopf bifurcations. This sample shows that fold-hopf bifurcations are more likely to occur when there is negative risk premium and bias towards home assets. We may also have scenarios, where $h > 0$ and $d < 0$, but d takes a small negative value. The vast majority of fold-hopf points occur when $h < 0$ and small. To get further insight in the possible transitions arising from the fold-hopf point, we perturbed each of the system parameters, $\tilde{\mu}$, by $\Delta\mu = \pm 0.05$, for the entire computed sample. For transitions where both solutions are consistent with $Z_1^{**} > 0$ and (25), we may have saddle-repellor and saddle-saddle scenarios. The first scenario suggests that unfoldings are simple for this case and relevant nonlinear phenomena is not a likely outcome. However, the saddle-saddle unfolding scenario may lead to complex dynamic phenomena, as a result of the complex organization of the saddle's invariant manifolds. The existence of a general *Hopf* bifurcations in the vicinity of these transitions, as

[27]The easiest path to detect a supercritical *Hopf* bifurcation in hyperbolic autonomous dynamical systems is to continue equilibrium from a spiral attractor. The numerical results described in section 4 suggest that this is not a likely outcome. The results obtained in this section confirm our conclusions regarding steady-state stability.

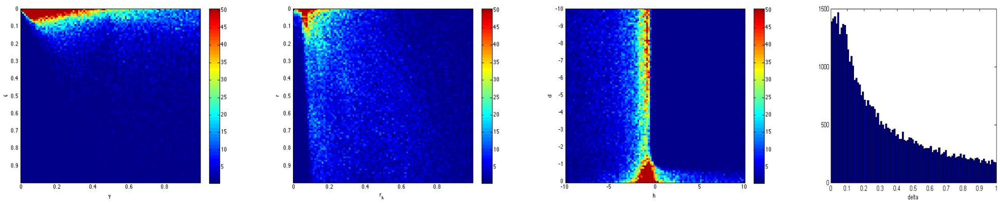
[28]This bifurcation is also called Gavrilov-Guckenheimer, saddle-node *Hopf*, zero *Hopf* and zero pair bifurcation. A detailed technical discussion of this bifurcation scenario can be found in Guckenheimer and Holmes [36] and Kuznetsov [47], chapter 7 section 4 and chapter 8 section 5, respectively.

[29]See Kuznetsov [47] page 339.

[30]Crommelin et al. [22] discusses heteroclinic cycles driving atmospheric regimes, Hilker and Malchow [38] portrays strange periodic attractors in population dynamics, and Izhikevich [42] shows bursting in neurocience dynamic systems in the vicinity of fold-hopf bifurcations.

[31]The parameter distributions for the fold-hopf bifurcation were computed numerically for a sample with 10^9 random draws, assuming parameters distributed uniformly, $\gamma, \delta, r \sim U(0, 1)$, $h \sim U(-10, 10)$ and $d \sim U(-10, 0)$. We considered fold-hopf co-dimension two outcomes consistent with $0 < r_k^{**}, \rho^{**} < 1$, $Z_1^{**}\left(r_k^{**}, \rho^{**}\right) > 0$ and (25).

previously discussed, may also play a role on the complexity of both unfolding scenarios. Still, the absence of attracting *Hopf* bifurcations in the vicinity of the unfolding, limits the range of nonlinear phenomena that may arise in this system. In section 7, we discuss with more detail some of the complex dynamics that may arise from this bifurcation, and put forward some conjectures and examples with meaningful economic interpretation.



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Histogram δ

Figure 2: Computed parameter density distributions for Fold-Hopf bifurcation

6. Economic space boundary dynamics

The existence of non feasible steady states in economic models has not shared the same amount of attention in literature, when compared to its economic counterparts. From a mathematical modeling point of view, the unjustified existence of these fixed points is sufficient ground to question the quality of a proposal. Since many economic models share this specific feature, we link the absence of a consistent discussion on this topic, to the lack of explanatory power of linear and quasi-linear proposals. In this section, we discuss the role of non feasible fixed points in a economic setting and derive policy rules that guarantee that orbits starting within the economic space, stay in this region. This concept is crucial, since it solves the modeling issue described, and introduces a novel policy objective. In Figure 3, we describe this mechanism schematically. Flows in the vicinity of the nullcline plane, $\dot{Z}_1(t) = 0$ for $Z_1(t) = 0$, are locally repelling for $Z_1(t) > 0$, when the growth rate of scaled consumption, $\dot{Z}_1(t)$, is positive in the vicinity of this separatrix plane. Economic recovery can be achieved by an endogenous dynamic mechanism if the necessary institutional framework is in place.

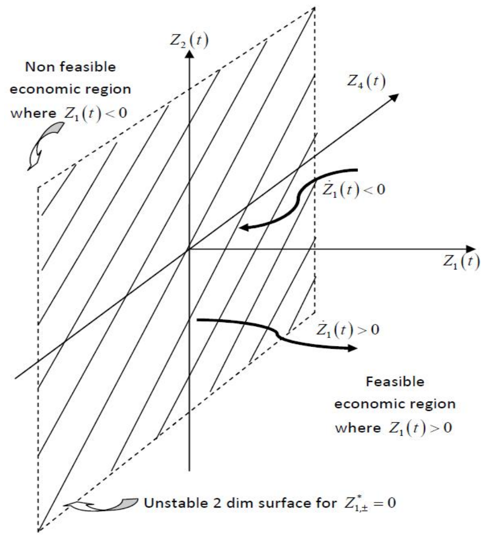


Figure 3: Separatrix planes for feasible economic regions

The mechanism described in Figure 3 has the following mathematical interpretation. Any trajectory, $\Delta(t)$, which starts or is in the vicinity of the region described by $Z_1(t) \approx 0 \wedge Z_1(t) > 0$, will stay in the space described by $Z_1(t) > 0$, if the following invariant condition is fulfilled, $\dot{Z}_1(t) = Z_1(t) f(Z_1(t), Z_2(t), Z_4(t)) > 0 \implies f(Z_1(t), Z_2(t), Z_4(t)) > 0$. This condition guarantees that the invariant plane, $\dot{Z}_1(t) = 0$ when $Z_1(t) = 0$, is locally repelling. Since the scaled consumption equation for this economy, (16), is already in the functional form of the invariant condition, a sufficient condition for the scenario described in Figure 3 is given by:

$$\frac{\rho - r(1 + dZ_2(t)) + (\delta - Z_4(t))(\gamma - 1)}{\gamma - 1} > 0. \tag{45}$$

However, the invariant condition given in (45), does not add much to our knowledge of the system. It just guarantees that trajectories starting in the economic space will stay there. In order to have a greater insight on the dynamics in the vicinity of this plane, we have to consider the qualitative dynamic properties of the nullcline surface dominating this region. For presentation purposes, we shall assume for now that a necessary condition for the existence of a repelling frontier, requires that the local dynamics of $\tilde{Z}_i^*$, have all at least an unstable dimension equal to two. Since in Appendix C, we are able to derive the general expressions in (C.7), for the eigenvalues describing local dynamics in the vicinity of $\tilde{Z}_i^*$. It is straightforward to define a set of rules that guarantee this outcome is fulfilled. Two scenarios can be considered, when we take into account the interaction between the institutional parameters, d and h. When a country faces an institutional framework consistent with $dh < 0$, the boundary has a unstable dimension equal at least to two when $J_{2,2}^* > 0$,

$$r + rdZ_2^* > Z_4^* - \delta. \tag{46}$$

This rule has two possible interpretations, when we to take into account the net financial status of an economy towards the rest of the world. In economies facing contraction and accumulation of foreign debt, $Z_2(t) > 0$, the growth rate of debt should be bigger than the growth rate of domestic capital. This rule allows for growth of domestic assets, as long as it occurs at a smaller rate than the interest growth on foreign bonds. If a country faces negative risk premium it might impose disinvestment on domestic assets. Recall that condition (46) only guarantees that local dynamics in this plane have an unstable dimension equal to two. To guarantee that solutions are repelled in all dimensions, the growth rate of consumption has also to be positive in the vicinity of the economic frontier, $J_{1,1}^* > 0$, following the result in (C.7). Since in the long run both foreign debt and domestic capital have to follow a balanced growth path, two hypothesis can be considered, in order to guarantee that the gap between growth paths does not widen in the long run. The first solution implies a contraction of the consumption growth rate and of the domestic capital accumulation rate, in order to reduce the level of debt. This hypothesis implies reduced investment and slower growth dynamics. The second hypothesis involves offsetting domestic capital returns to allow for investment and consumption growth to catch up with the faster debt growth rate. This scenario is consistent with challenges posed to economies with current balance deficits and facing contraction due to excess debt service. In economies facing contraction and accumulation of foreign assets, $Z_2(t) < 0$, the growth rate of investments abroad might impose disinvestment in domestic assets. In broader terms, this rule implies that policy has to guarantee that the rate of growth of revenues on foreign asset from domestic investment abroad, cannot be used exclusively on domestic capital accumulation. These foreign capital influxes have to finance domestic consumption. When an economy faces positive risk premium it might impose disinvestment on domestic capital. This rule provides an indirect instrument guaranteeing that foreign based capital revenues are used in domestic consumption activities during contractions. In the long run, growth stability has to achieved by an increased growth rate of consumption and domestic capital. This scenario is consistent with challenges posed to economies with current balance surplus facing contractions due to reduced world demand for their goods and services. In economies facing an institutional framework consistent with, $dh > 0$, two policy rules have to be considered:

$$J_{2,2}^* > 0 \wedge J_{2,2}^* > J_{2,3}^* \sqrt{\frac{rd}{h}}; \tag{47}$$

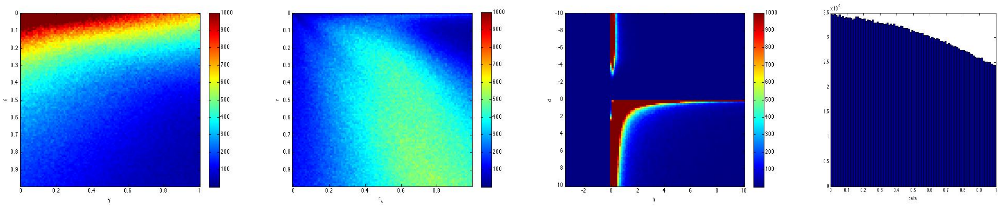
$$J_{2,2}^* < 0 \wedge J_{2,2}^* < J_{2,3}^* \sqrt{\frac{rd}{h}} \Longrightarrow J_{1,1}^* > 0. \tag{48}$$

The rule described in (47) has the same interpretation as the rule described in (46). Local dynamics in the boundary surface are only repelling when $J_{1,1}^*, J_{2,2}^* > 0$ holds. The interpretation for creditor and debtor economies given in the previous paragraph, still holds for this case. When the constraint, $J_{2,2}^* > 0$, does not hold, the only policy solution available is to guarantee that the growth rate of consumption is always positive, as described in (48). This is a last resort option. The policy-maker has to guarantee consumption growth in the event of severe institutional environment, $d, h > 0$, as observed in economies facing balance of payment crisis leading to exchange rate crisis. In such cases, only direct intervention to curb consumption dynamics guarantees that local dynamics in the vicinity of $\tilde{Z}_i^*$ have at least two unstable dimensions. In this scenario, it is not guaranteed that trajectories are repelled in all dimensions near the economic frontier, as non feasible fixed points have at least one stable dimension.

We finish this section with the sampling results obtained for local dynamics satisfying the rules described in (46) to (48). In Figure 4, below, we portray the parameter distributions consistent with these rules. [32]. A quick

[32] The results portrayed in this section were obtained from a sample with 10^9 random draws of uniform distributed parameter, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$, $d, h \sim U(-10, 10)$. The numerical computation of steady-states followed the definitions given in Appendix B. Only computed steady-states consistent with a maximum absolute error smaller than 10^{-5} were considered.

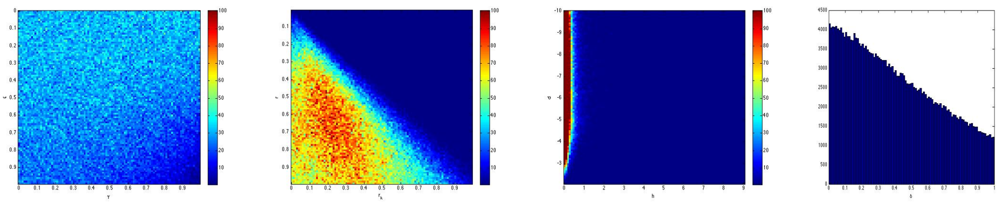
inspection shows that the existence of local dynamics with an unstable dimension equal to two is more likely to occur when $h, d > 0$. It is also a likely outcome for institutional scenarios described by $d < 0$ and $h > 0$. In section 2, we related these institutional scenarios with economies facing dire economic conditions. In our opinion, this result has the following interpretation. This nonlinear setup is capable of capturing the existence of a dynamic mechanism that avoids ever declining economic trajectories for countries facing severe institutional and financial conditions. Economies with favourable institutional frameworks do not require the existence of this dynamic mechanism.



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

Figure 4: Computed parameter densities for unstable dimension two frontier regions

When we consider the sample described in Figure 4 and assume that $\mathrm{Re}\left(\Lambda^*_{1,2,3}\right) > 0$, is fulfilled for all $\tilde{Z}^*_i$, two patterns arise. First, local repelling dynamics in the economic boundary are only consistent with $d < 0$ and $h > 0$. In section 2, we related this institutional scenario to foreign interventions guaranteeing a temporary negative risk premium environment, for countries facing capital flights due to dire domestic financial conditions. The results in Figure 5 seem to support foreign policy interventions that guarantee a temporary debt subsidy to distressed nations. Second, sampling results suggest that returns on domestic assets have to be smaller than the international interest rate, $r_k < r$. We conclude that a dynamic recovery path may exist for countries facing productivity problems, as long as they are able to access foreign capital at a sustainable level. Finally, recall that this set of rules only guarantees that flows are repelled when they approach the economic frontier. Convergence to a balanced growth regime depends on additional factors. If these are not met, there is a risk that boundary interactions result in explosive debt dynamics and create an unsustainable economic environment.



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

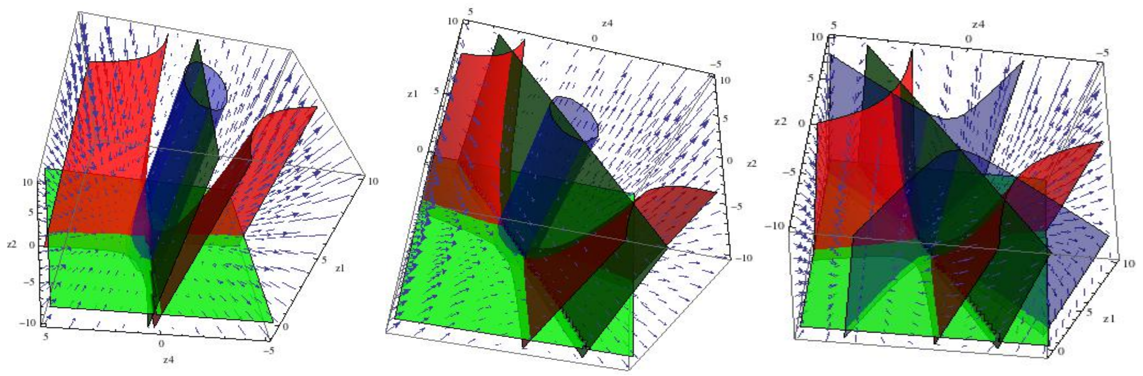
Figure 5: Computed parameter densities for repelling frontier regions

7. Global dynamics: Conjectures, examples and policy implications

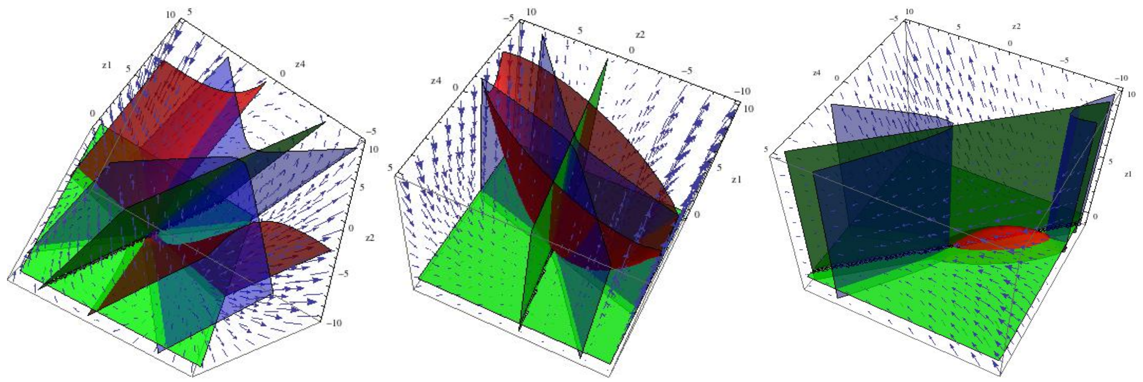
We conclude this presentation with a discussion on the global dynamics[33] of this system. We start with a description of the phase-space organization for $Z_1(t) > 0$. In the previous section, we discussed the local dynamics of the nullcline plane $\dot{Z}_1(t) = 0$ for $Z_1(t) = 0$. When $Z_1(t) > 0$, the $\dot{Z}_1(t) = 0$ nullcline defines another plane that intersects the boundary plane given by $Z_1(t) = 0$. The intersection of these planes is given by the line described by the steady-state expression (35). The remaining nullclines are described by two quadrics. In Appendix E, we classify geometrically these quadratic surfaces and show that the phase space organization depends on institutional scenarios for h and d combinations. At $hrd = 1$ the phase-space simplifies dramatically. For the remaining cases we distinguish between six relevant scenarios. These are described in Figure 6 below.

The scenarios depicted in(6) portray the challenge posed to the policy-maker in a nonlinear environment. Different institutional setups impose drastic changes in the phase space organization. Further, the absence of local stable solutions, implies that the existence of asymptotic orbital stable solutions, for a given institutional scenario,

[33] A detailed mathematical introduction to the topic of global dynamics for flows in $\mathfrak{R}^3$ can be found in Wiggins [87] and Wiggins [88].



(a) $h, d < 0 \wedge 0 < hrd < 1$ (b) $h, d > 0 \wedge 0 < hrd < 1$ (c) $d < 0 \wedge h > 0$



(d) $d > 0 \wedge h < 0$ (e) $h, d < 0 \wedge hrd < 0 \wedge hrd > 1$ (f) $h, d > 0 \wedge hrd < 0 \wedge hrd > 1$

Figure 6: Phase space organization for different institutional scenarios

requires the existence of an attractive set[34]. Alternatively, we can assume that the policy-maker challenge is a boundary value problem, more specifically a *Turnpike* control problem. *Turnpike* theory has its roots in modern growth theory. For optimal growth models, McKenzie [57] frames the policy problem[35], as one of finding the fastest route to the desired solution, when the departure point is far from the final long run solution[36]. In our setup, the *Turnpike* problem reduces to a problem of placing orbits on the stable manifold of a saddle solution that fulfils $\max\left(Z_1^{**}\right)$ and (25). Initial values for the two controls variables, $Z_1(0)$ and $Z_4(0)$, can be chosen for this purpose. Discontinuous jumps of the control variables for $t > 0$ can also be considered. However, several issues arise with this approach in a nonlinear environment with multiple equilibria. First, it is unlikely that the stable manifolds of optimal saddle solutions can be computed exactly. This is particularly true when $d > 0$, since feasible solutions have a stable dimension equal to one. A realistic option is to shoot trajectories towards the boundary saddle value solution and take into account system dynamics in the vicinity of the saddle stable manifold. This is a technically feasible task but of difficult application for nonlinear vector fields in $\Re^3$. The literature on this subject suggests orbital control at a value loss. In discounted problems the quality of this method worsens as the turnpike distance increases. This is a result of the value loss boundaries widening as $t \to \infty$. Second, in the event of a small parameter perturbation for $t > 0$ and $Z_i(t) = Z_i^{**}$, the *Turnpike* solution may no longer be the best and/or an optimal boundary solution to the control problem. In this context, a *Turnpike* control policy: (i) imposes a discontinuous jump towards the stable manifold of the new best optimal saddle solution, and we have a *Turnpike* heteroclinic connection of equilibria; or (ii) the parameter perturbation leads to the disappearance of optimal saddle solutions and the sole policy option available relies on the existence of a *Turnpike* path towards an attractor solution asymptotically consistent with (25). This second hypothesis stresses the need to evaluate the existence of attractive sets driven by global interactions in nonlinear multi-equilibria growth models. As flows bounded by hyperbolic compact sets are likely to arise in the vicinity of phase space singularities, the analysis of bifurcations is crucial to understand the specific nonlinear global phenomena that may arise in this system.

[34] By attractive set we refer to the broad definition of an attractor, where flows starting in the neighbourhood of the attractive set, called the attractor basin, asymptotically evolve towards an invariant closed subset of the phase space. This invariant set is the attractive set.

[35] For some recent developments and open problems in *Turnpike* theory and optimal growth see McKenzie [58].

[36] If the origin is in the vicinity of the final solution and the *Turnpike* far away, then the best policy option may not involve the *Turnpike*.

To demonstrate our arguments, we start by considering two conjectures. We evaluate the conditions for the existence of heteroclinic and homoclinic dynamics consistent with the definition of attractive sets. In this setup, heteroclinic orbits[37] correspond to flows connecting long run growth regimes, while homoclinic orbits[38] can be linked to temporary structural change dynamics. Recall that in section 1, we introduced the concept of endogenous structural change, as a valid hypothesis to explain the structural breaks observed in macroeconomic data, and portrayed this empirical phenomena in Figure 1, for US and UK log (GDP) time series. We related this hypothesis to the existence of heteroclinic and homoclinic phenomena. However, heteroclinic and homoclinic orbits are not consistent with the strict definition of structurally stable solutions[39]. In general terms, the strict structural stability criterion imposes that the qualitative features of a system are robust under small parameter perturbations[40]. This criterion imposes severe limitations to the study of nonlinear global phenomena, since solutions defined by attractive sets may be asymptotically, but not structurally, stable under small perturbations. As Guckenheimer and Holmes [36] puts it: *"This principle was embodied in a stability dogma, in which structurally unstable systems were regarded as somehow suspect. This dogma stated that, due to measurement uncertainties, etc., a model of a physical system was valuable only if its qualitative properties did not change with perturbations."*. The authors suggest a redefinition of the structural stability paradigm[41] that takes into account the complexity of global nonlinear phenomena. *"Thus the stability dogma might be reformulated to state that the only properties of a dynamical system which are physically relevant are those which are preserved under perturbations of the system. The definition of physical relevance will clearly depend upon the specific problem."*. Taking into account this broader definition, we argue that a reasonable criterion for relevant solutions is, asymptotic orbital stability under small perturbations consistent with optimal growth dynamics. Even if these perturbations lead to qualitative changes and transitions between attractors. In a policy framework our argument has the following interpretation. The policy-maker should acknowledge the complexity of interactions driving the short run economic process and decisions should be restricted to policies that promote a long run stable growth environment. Even if this *laissez faire* approach results in the economy undergoing structural changes in the short run.

A conjecture consistent with the above criterion is the heteroclinic cycle scenario. This hypothesis has an interesting economic interpretation and introduces novel challenges to macroeconomic policy definition. In our setup, the interaction between saddles, with different stable dimensions, may be consistent with the existence of heteroclinic cycles. In section 5, we referred how the unfolding of a fold-hopf bifurcation might be consistent with this phenomena[42]. Heterocilinic cycles in this context arise from homoclinic bifurcations and are preceded by chaotic parameter regimes. Long run growth dynamics driven by heteroclinic cycles are characterized by long lasting fast growing regimes that undergo increasing, and then decreasing periods of volatility, before a crisis event drives the economy abruptly to the slow growth regime of the past. Evolutionary growth theories suggest that severe crisis, or the downturn of the long wave cycle, is preceded by fast growth regimes with low volatility. In a recent article on the *Great Moderation*, the 2007-2008 financial crisis and the resulting strong economic contraction, Bean [9] suggests that: *"The longer the low volatility period lasts, the more reasonable it is to assume that it is permanent. But as tail events are necessarily rarely observed, there is always going to be a danger of underestimating risks"*.

[37] An orbit, $\Delta(t)$, is said to be heteroclinic if it connects two fixed points, Z_a^{**} and Z_b^{**}, such that: (i) $\Delta(t) \longrightarrow Z_b^{**}$ as $t \longrightarrow +\infty$, and (ii) $\Delta(t) \longrightarrow Z_a^{**}$ as $t \longrightarrow -\infty$. Where Z_b^{**} is a stable feasible solution that fulfils (25).

[38] An orbit, $\Delta(t)$, is said to be homoclinic if $\Delta(t) \longrightarrow Z_a^{**}$ as $t \longrightarrow \pm\infty$. Where Z_a^{**} is a economic meaningful equilibrium of our system that fulfils (25). Homoclinic orbits are characterized by having a infinite period but finite length. In layman's terms, this means that diverging flight trajectories eventually converge to the stable manifold of the saddle equilibrium, where they will stay longer and longer, before starting another flight. The invariant set describing homoclinic flows can thus be asymptotically approximated to Z_a^{**}.

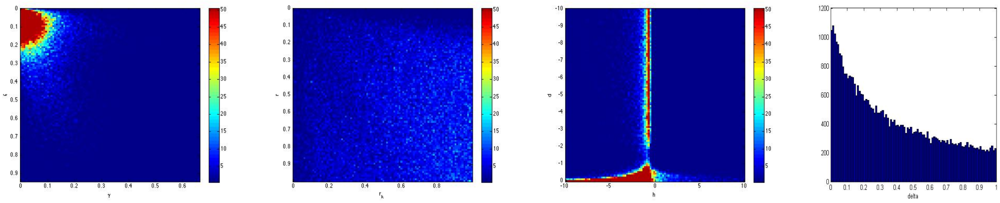
[39] Structural stability is a fundamental concept of dynamic systems theory. It was introduced by the russian mathematicians Aleksandr Andronov and Lev Pontryagin. A formal proof of *Andronov-Pontryagin* structural stability criterion exists only for vector fields in $\Re^2$. The proof is given by Peixoto [65] and Peixoto [66].

[40] There are many extensions of the strict mathematical definition of structural stability in applied nonlinear science. It is common to find proposals that define structural stability following the *Andronov-Pontryagin* strict criterion, but take into account model specificities and theoretical implications of different structural stability definitions, to provide a consistent measure of the structural stability of a given system. Some examples of this approach can be found in economic literature. Fuchs [29] discusses the implications for economics of the notion of structural stability. Anderlini and Canning [2] propose bounded rationality as a criterion of structural stability in dynamic games with fully rational players. Zhang [90] emphasizes the need for a structural dynamic approach in economics and discusses possible implications of complexity theory for the study of economic processes.

[41] The structural stability dogma also played a role on the development of economic theory. The neoclassical critic on the evolutionary economic paradigm has been based on the grounds that evolutionary economic models are inherently structurally unstable. In a recent article, Veneziani and Mohun [85], reviews the neoclassical critic of Goodwin's growth cycle model and evolutionary dynamics approach. The author suggests that the neoclassical dismissal of evolutionary dynamic proposals on these grounds, is not in accordance with the modern mathematical concept of structural stability. In a recent working paper, Matteo [56] reviews early discussions on structural stability and economic dynamics by Morishima and discusses its implications for neoclassical growth theory, following *Solow's* seminal proposal on economic growth.

[42] The interaction of the stable and unstable dimensions of the two saddles creates a compact set with a sphere geometry, which results in dense orbits connecting the two equilibrium. See Crommelin et al. [22] for a clear geometric description and discussion of this phenomena in the vicinity of fold-hopf bifurcations.

According to the author, the forecasting problem faced by the decision maker is exacerbated by the lack of information that is required to learn the higher moments of economic distributions. In other words, the complexity of nonlinear phenomena poses dramatic challenges to the management of macroeconomic risks. To evaluate the feasibility of this conjecture in our setup, we consider a conservative scenario, where Z_a^{**} and Z_b^{**} both fulfil (25) and $Z_1^{**} > 0$. Below, in Figure 7, we portray the parameter distributions consistent with this conjecture[43].



(a) Density ρ, γ (b) Density r, r_k (c) Density d, h (d) Prob. δ

Figure 7: Computed parameter densities for saddle-saddle optimal scenarios

Saddle-saddle interactions are more likely to occur when $h, d < 0$, but may also occur when $h > 0, d < 0$, for small values of h and d. We conclude that meaningful heteroclinic cycle scenarios are only likely to occur in the vicinity of fold-hopf bifurcations, following the results portrayed in Figure 2.

The second conjecture proposed, is related to the structural change phenomena observed in Figure 1, for US $\log(GDP)$ data. In this example, the estimation procedure computed two breaks for the second quarter of 1978 and the last quarter of 1983. The difference between the estimated slope shifts is approximately zero. We extrapolate that a temporary medium run crisis led the economy out of its long run growth path, into a diverging and then converging flight, which eventually rested in the initial long run growth equilibrium. This dynamic event is consistent with the mathematical definition of a homoclinic orbit. In this scenario, an economy will stay long periods in the vicinity of long run equilibrium, but will undergo crisis or hysteria for short periods of time. Homoclinic phenomena has been gaining attention in recent growth literature. Benhabib et al. [11] and Mattana et al. [55], for example, evaluate homoclinic bifurcations in continuous time endogenous growth models. However, homoclinic orbits are not robust to small perturbations. To overcome this issue, we focus on a conjecture consistent with the existence of *Sil'nikov* homoclinic phenomena. We had already mentioned in section 5 that this scenario may occur in the vicinity of fold-hopf bifurcations. The original *Sil'nikov* scenario is a basic criteria for system complexity, where strange attractors are born from transitions from a homoclinic bifurcation [44]. The *Sil'nikov* bifurcation can be described in the following fashion. If the leading eigenvalue condition is fulfilled in the vicinity of a saddle focus homoclinic bifurcation, then trajectories diverge faster along the one dimensional outset than the convergent trajectories along the two dimensional inset. In the vicinity of this parameter region there is a transition where orbits generated by the system become increasingly more complex homoclinic loops and by definition represent dense solutions to the system[45]. We evaluate under what conditions saddle focus solutions fulfil the leading eigenvalue condition. The parameter distributions consistent with this conjecture are given below in Figure 8[46].

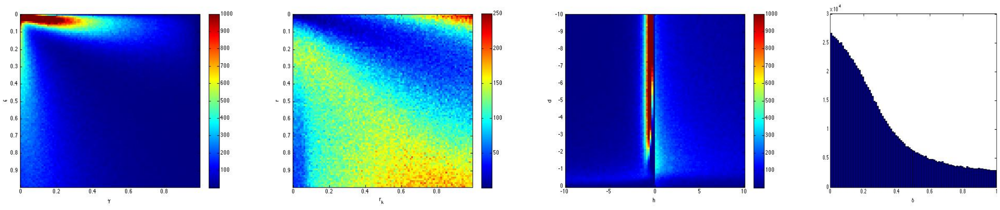
As in the previous example, sample results suggest that this scenario is more likely to occur in the vicinity of fold-hopf bifurcations, when $d < h < 0$, and may also occur when $d < 0$ and $h > 0$. A closer look at the h, d density plots in Figure 8 and Figure 7 reveals a pattern consistent with the joint distribution described in Figure 2 for the fold-hopf sample. To confirm the existence of *Sil'nikov* homoclinic scenarios, the application of numerical continuation methods and the simulation of orbits using normal forms, is required. The computed

[43] The results were obtained using our stochastic sampling method, from a sample with 10^9 random draws, assuming parameters distributed uniformly, $\rho, \gamma, \delta, r, r_k \sim U(0,1)$, $d, h \sim U(-10, 10)$.

[44] Other *Sil'nikov* scenarios can be considered for this system. Piscitellia and Sportelli [68] demonstrate the existence of inverted *Sil'nikov* homoclinic phenomena in a nonlinear continuous version of the inventory cycle model. This scenario involves divergence in the two dimensional outset and convergence in the one dimensional inset of a saddle focus solution. The leading eigenvalue condition is given by $\left|\Lambda_1^{**}\right| < Re\left(\Lambda_{2,3}^{**}\right)$. This bifurcation scenario was originally proposed by Tresser [80]. The coexistence of saddle focus solutions of different stable dimensions also suggests the possibility of multi-circuit *Sil'nikov* homoclinic dynamics, following the proposal by Gaspard [30]. See Piscitellia and Sportelli [68] for a demonstration of this phenomenon in an economic setup.

[45] Sufficient conditions for the existence of saddle focus homoclinic bifurcations require non-degeneracy, inclination and orbit properties to be fulfilled. The leading eigenvalue condition is given by, $\Lambda_1^{**} > \left|Re\left(\Lambda_{2,3}^{**}\right)\right|$. A recent survey and detailed mathematical discussion on this topic can be found in Homburg and Sandstede [39].

[46] The results portrayed follow the same specifications of the previous example. We check for saddle focus solutions consistent with (25), by computing first the discriminant of J^{**}, Γ, and then checking if necessary conditions, $\Gamma < 0$, $Re\left(\Lambda_{2,3}^{**}\right) < 0$ and $\Lambda_1^{**} > \left|Re\left(\Lambda_{2,3}^{**}\right)\right|$, are fulfilled.

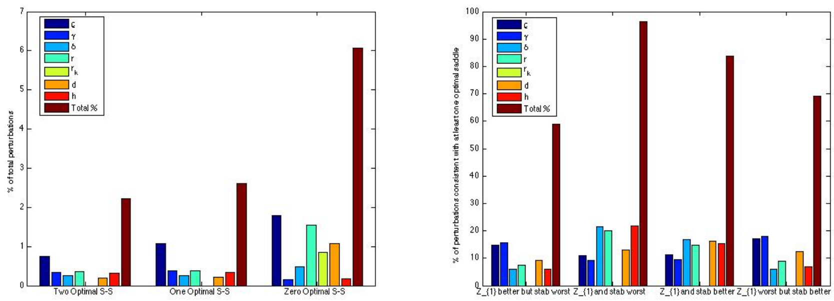


(a) Density ρ, γ (b) Density r, r_k (c) Density d,h (d) Prob. δ

Figure 8: Computed parameter densities for saddle focus consistent with *Sil'nikov* scenario

fold-hopf bifurcation sample provides an interesting starting point. However, one word of advice, performing such task is comparable to searching for a needle in the haystack. Fold-hopf points may undergo transitions consistent with this scenario, but other transitions are possible. Given the amount of bifurcation points computed and the vast parameter space, the choice of an optimal candidate for numerical bifurcation analysis is not an obvious decision.

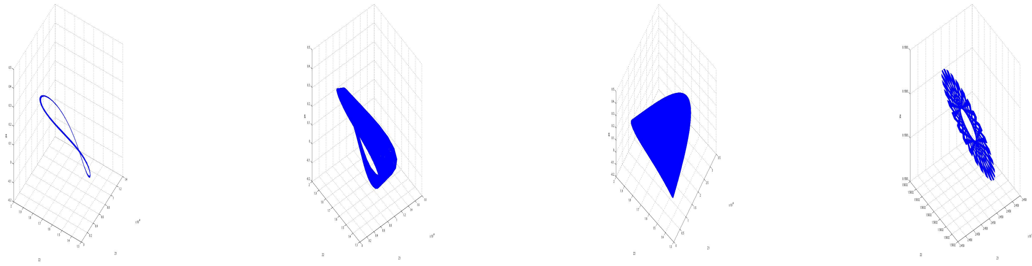
Recall that in the beginning of this section, we referred to the implications of considering *Turnpike* control solutions in a multiple equilibria setup, when small parameter perturbations are considered. In Figure 7 we portrayed the parameter samples describing the co-existence of optimal saddle solutions. To test the hypothesis of heteroclinic connections of equilibria arising from *Turnpike* control dynamics, we evaluated the saddle-saddle sample and checked for qualitative and quantitative changes induced by small parameter perturbations. Figure 9, below, portrays the results obtained. For simplicity reasons, we now assume that Z_a^{**} and Z_b^{**} are the initial saddle solutions and $Z_a^{**} > Z_b^{**}$, while $Z_a^{**}(\Delta\mu)$ and $Z_b^{**}(\Delta\mu)$, are the resulting steady-states assuming a parameter perturbation equal to $\Delta\mu = \pm 0.01$. The figure on the left portrays the quantitative sensitivity of Z_a^{**} and Z_b^{**} to parameter perturbations as a percentage of the total perturbations. We considered three cases of interest: (i) on the left we have $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}(\Delta\mu) > 0$ and $\delta < Z_{4,a}^{**}(\Delta\mu), Z_{4,b}^{**}(\Delta\mu) < \delta + \rho/\gamma$; (ii) on the center we have $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}(\Delta\mu)$, $Z_{1,b}^{**}(\Delta\mu) > 0$ and $\delta < Z_{4,b}^{**}(\Delta\mu) < \delta + \rho/\gamma$, while $Z_{1,a}^{**}(\Delta\mu) < 0$ and/or $Z_{4,a}^{**}(\Delta\mu)$ does not fulfil (25); and (iii) on the right $Z_{1,a}^{**}(\Delta\mu), Z_{1,b}^{**}(\Delta\mu) < 0$ and/or $Z_{4,a}^{**}(\Delta\mu), Z_{4,b}^{**}(\Delta\mu)$ does not fulfil (25). We then focused on the first two cases and checked their qualitative outcomes. The figure on the right describes the results obtained. From left to right, the four cases of interest are described by: (i) $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with one stable dimension; (ii) $Z_{1,b}^{**}(\Delta\mu) < Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with one stable dimension; (iii) $Z_{1,b}^{**}(\Delta\mu) > Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with two stable dimensions; and (iv) $Z_{1,b}^{**}(\Delta\mu) < Z_{1,a}^{**}$ and $Z_{1,b}^{**}(\Delta\mu)$ is a saddle with two stable dimensions.



(a) Parameter sensitivity (b) Best saddle outcomes

Figure 9: Parameter perturbations leading to best saddle switch and best saddle qualitative outcomes

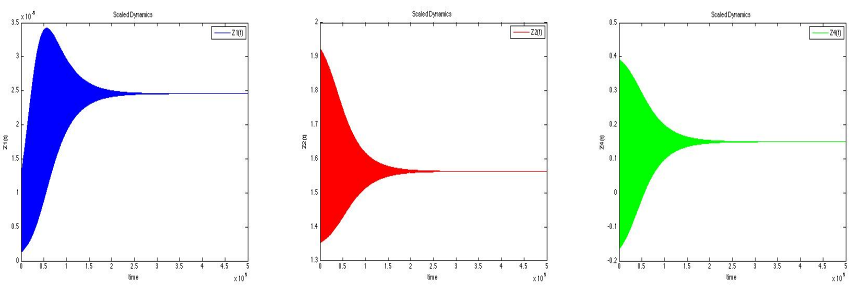
The results portrayed in Figure 9 show that there are parameter regimes where complex outcomes may also arise from *Turnpike* control dynamics. If *Turnpike* dynamics are not able to impose heteroclinic connection paths of equilibria, then a small parameter perturbation may throw the economy into a low growth regime, or worst, into a non-optimal growth regime. Moreover, there are parameter regimes, where small perturbations lead to a phase-space organization, where orbits on the stable manifold of a saddle are no longer consistent with the notion of optimal growth dynamics. In this specific case, optimal dynamics may only be feasible in the vicinity of attractors arising from the complex organization of the invariant manifolds. This result stresses the importance of analysing global dynamics in economic systems. To illustrate this argument, we finish this discussion with an example of a parameter regime consistent with complex global phenomena. For this purpose, we consider a non orthodox parameter set, in the vicinity of a fold-hopf point, where $d < 0$ and $h > 0$. The simulated orbit is portrayed below, in the left picture of Figure 11. This flow[47] has a limit cycle behaviour and fulfils (25). For this combination of parameters, there is only a feasible, but non optimal, steady-state solution. We conclude that in this phase space region, orbits are driven by the complex organization of the feasible and boundary fixed points manifolds. This solution is not structurally stable. When a small perturbation is imposed, $\Delta\gamma = 0.0001$, trajectories are attracted to another region of the phase space. This dramatic transition is portrayed by the center pictures of Figure 10. The last figure shows the asymptotic dynamics for this transition.



(a) Optimal cycle dynamics. (b) Initial transition (c) Full transition dynamics (d) Asymptotic dynamics

Figure 10: Optimal global dynamics and transition of a structurally unstable solution

The asymptotic behaviour of this flow is characterized by small amplitude limit cycle dynamics. This behaviour suggests that both repelling and attracting forces are at work in this region. We are not able to confirm if this orbit fulfils (25) asymptotically, since $\langle Z_4 \rangle = 0.1502 \approx \delta + \rho/\gamma = 0.1502$. This attractive set is robust to a wide range of perturbations, but it is not sensitive to the sign of the parameter variation. For example another transition to a different attractor occurs for small perturbations of d or h, if $\Delta d, \Delta h < 0$. Given that numerical integration routines are not able to correctly capture system dynamics near complex singularities, we cannot rule out the possibility of further nonlinear phenomena, such as bursting, which is known to arise in the vicinity of fold-hopf bifurcations. The time series describing the variables transition from the initial flow are given below in Figure 11.



(a) $Z_1(t)$ dynamics (b) $Z_2(t)$ dynamics (c) $Z_4(t)$ dynamics

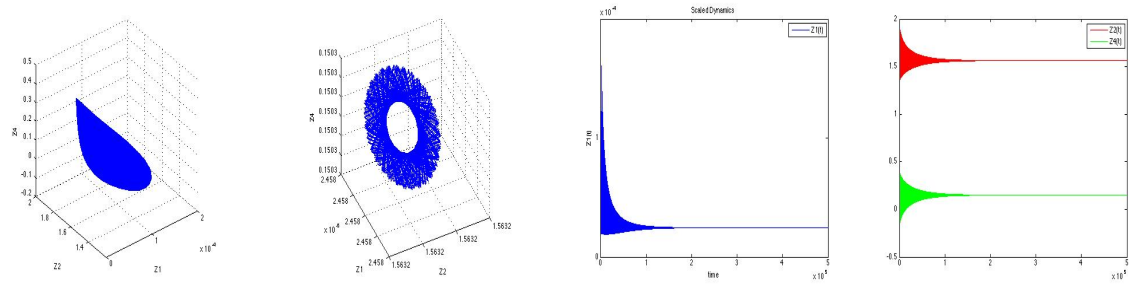
Figure 11: Scaled dynamics during transition

The dynamics portrayed in Figure 11 show an over-indebted economy undergoing structural change, where initial investment dynamics are extremely volatile and domestic capital growth rates alternate between expansion and contraction periods. This behaviour is a result of the initial high level of debt. Investment expansions depend on foreign capital flows, which further increases the debt load until it becomes unsustainable and investment has to contract. As the economy accumulates productive capital, investment volatility decreases and the economy settles in a long run regime with small amplitude cycles. The structural change occurring in this economy is portrayed by the dynamics of consumption. Consumption volatility increases during the transition period, before decreasing and settling in the small amplitude long run cycle growth regime.

Finally, we evaluated the basin of attraction for this attractor and confirmed that it holds for a broad range of $Z_2(0)$ and $Z_4(0)$ values. However, it is sensitive to small perturbations of $Z_1(0)$. We had already mentioned

[47] We integrate all the orbits using a Runge-Kutta of the $8^{th} - 7^{th}$ order and set the relative and absolute error tolerance to 10^{-10}. This orbit is obtained for the parameter set: $\rho = 0.099704$, $\gamma = 0.731579$, $\delta = 0.013929$, $r = 0.892695$, $r_k = 0.747145$, $d = -0.542038$, $h = 0.506959$, given initial conditions: $Z_1(0) = 0.000007$, $Z_2(0) = 1.921564$, $Z_4(0) = 0.150233$, where $\langle Z_4 \rangle = 0.133 < \delta + \rho/\gamma = 0.1502$, consistent with (25). The fold-hopf point is: $\rho = 0.099704$, $\gamma = 0.731479$, $\delta = 0.013929$, $r = 0.893695$, $r_k = 0.947145$, $d = -0.441038$, $h = 0.6812$.

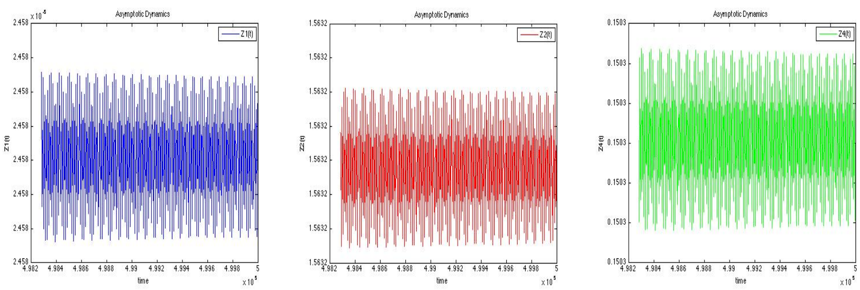
that the behaviour of this attractor is sensitive to the sign of parameter perturbations and that orbits may converge to another attractive set. We demonstrate this transition by imposing a small variation on initial consumption, $\Delta Z_1(0) = 0.0001$. The dynamics of this transition are portrayed below in Figure 12. The two figures on the left portray the phase space transition and asymptotic dynamics, respectively. The two figures on the right show the time series obtained for this transition.



(a) Full transition dynamics (b) Asymptotic dynamics (c) $Z_1(t)$ dynamics (d) $Z_2(t), Z_4(t)$ dynamics

Figure 12: Transition to second attractor due to sensitivity in initial conditions

Although the convergence process is somewhat similar for both attractive sets. The slightly different departure point sends the economy to a region with a lower consumption level, as investment volatility decreases. Asymptotic dynamics for this case are consistent with (25), $\langle Z_4 \rangle = 0.1495 < \delta + \rho/\gamma = 0.1502$. The phenomena portrayed by these two transitions, illustrates the challenges faced by policy-makers in economies facing dire institutional and financial conditions. We describe this challenge in the following fashion. There is a path towards expansion and stronger long run growth dynamics. However, the probability of the economy converging to this growth regime is low. If the ideal conditions are not met, it is more likely that the economy spirals down to the low growth regime. Finally, we cannot exclude further nonlinear phenomena arising from this transition, for the same reasons described previously. Asymptotic dynamics, portrayed in Figure 13, are now consistent with small amplitude quasi-periodic motion. It is also possible that these two attractive sets correspond to a sole attractor, or flows cycle between the two attracting regions. A more in depth analysis of this phase space region is required to be able to shed some light in these hypotheses.



(a) $Z_1(t)$ asymptotic dynamics (b) $Z_2(t)$ asymptotic dynamics (c) $Z_4(t)$ asymptotic dynamics

Figure 13: Asymptotic scaled dynamics

8. Conclusions and further research

In this paper we proposed a simple endogenous growth model, where asymptotic orbital stable solutions are only feasible for attractive sets arising from global interactions of multiple equilibria. We also described how the assumption of dynamic solutions defined in a *Turnpike* control setting is not sufficient to accommodate the complexities that may arise from this setup. We show that these outcomes are the natural consequence of considering multiple nonlinear mechanisms. This conclusion has several implications for policy in a dynamic setting. First, the analysis of local bifurcations in multiple equilibria setups, is crucial for the definition of policy rules in nonlinear environments. Second, the evaluation of global conjectures allows for a broader perspective of the challenges faced by a particular economy. Using a stochastic sampling method, we were able to map effectively the parameter space describing the institutional conditions for the existence of specific local phenomena and relevant global conjectures. We emphasize two main results from this analysis. When economies face a positive risk premium, the existence of optimal growth outcomes is limited to a saddle solution with only one stable dimension. This result suggests, as expected, that risk premium on sovereign debt plays a crucial role on the long run financial stability of an economy. This outcome could also be interpreted in another fashion. The capacity of the policy-maker to impose financial repression policies, consistent with real negative interest rates, is crucial to guarantee a stable financial and long run growth environment. In a recent working paper, Reinhart and Sbrancia [72] suggests an important role of financial repression and moderate inflation policies on the post war deleveraging period and subsequent decades of financial stability in developed western economies. Drelichman and Voth [25] gives historical evidence on this phenomenon and suggests that the British Empire capacity to overspend its rivals during the eighteenth century expansion period was linked to financial repression policies. The author compares this result with the decline of the Spanish Empire during the sixteenth century, which resulted in a series of defaults on Spanish sovereign debt. Our analysis of the phase space boundaries suggests that in the event of capital flights, due to severe economic conditions, policies capable of maintaining a negative risk premium, such as bilateral debt subsidies, are crucial for the existence of a dynamic recovery path. This result is in accordance with both economic theory and the modern policy paradigm. However, our model suggests that negative risk premium is only a necessary condition for recovery. The existence of a recovery consistent with convergence to a long run growth path requires that further institutional conditions are met. Finally, sampling results suggest that asymptotic orbitally stable solutions driven by complex global dynamics, and consistent with the definition of an attractor, are only likely to occur for institutional scenarios in the vicinity of fold-hopf bifurcations. This result has several implications for policy, as the unfolding of fold-hopf bifurcations has the potential to unleash a cascade of complex global dynamic events, and the full bifurcation scenario is still not fully understood. We give an example of the complex outcomes that may arise in this system, for an economy facing dire financial and institutional conditions, and describe the challenges posed to policy in this environment. To shed some light on this and other possible complex nonlinear phenomena, a thorough analysis based on modern numerical bifurcation analysis techniques is still required.

Finally, it is possible to scale this system and reduce the parameter space by assuming a translation to the center of the quadratic nullclines. The analysis of the resulting reduced system might provide important clues about the global organization of the phase space. We are aware of this hypothesis but leave this exercise to a future discussion.

Appendix

Appendix A. Optimal control conditions

The current value *Hamiltonian* for the intertemporal maximization problem given in (9) is,

$$H^{*}\left[C(t), I(t), B(t), K(t), \lambda(t), q(t)\right] = C(t)^{\gamma} + \lambda(t)\dot{B}(t) + q(t)\dot{K}(t), \tag{A.1}$$

where $\dot{B}(t)$ and $\dot{K}(t)$ are given in (A.6) and (A.7). The *Pontryagin* necessary and sufficient conditions for the existence of an optimum solution for (9), are given by:

Optimality conditions

$$\frac{\partial H^{*}}{\partial C(t)} = 0 \iff \gamma C(t)^{\gamma-1} = -\lambda(t); \tag{A.2}$$

$$\frac{\partial H^{*}}{\partial I(t)} = 0 \iff q(t) = -\lambda(t)\left(1 + h\frac{I(t)}{K(t)}\right); \tag{A.3}$$

Multiplier conditions

$$\frac{\partial \lambda(t)}{\partial t} = \rho\lambda(t) - \frac{\partial H^{*}}{\partial B(t)} \iff \dot{\lambda}(t) = \lambda(t)\left(\rho - r - rd\frac{B(t)}{K(t)}\right); \tag{A.4}$$

$$\frac{\partial q(t)}{\partial t} = \rho q(t) - \frac{\partial H^{*}}{\partial K(t)} \iff \dot{q}(t) = q(t)(\rho+\delta) + \lambda(t)\left[\frac{rdB(t)^{2}}{2K(t)^{2}} + \frac{hI(t)^{2}}{2K(t)^{2}} + r_{k}\right]; \tag{A.5}$$

State conditions

$$\frac{\partial B(t)}{\partial t} = \frac{\partial H^{*}}{\partial \lambda(t)} \iff \dot{B}(t) = C(t) + I(t)\left[1 + \frac{hI(t)}{2K(t)}\right] + rB(t)\left[1 + \frac{dB(t)}{2K(t)}\right] - r_{k}K(t); \tag{A.6}$$

$$\frac{\partial K(t)}{\partial t} = \frac{\partial H^*}{\partial q(t)} \Longleftrightarrow \dot{K}(t) = I(t) - \delta K(t); \tag{A.7}$$

Transversality conditions

$$\lim_{t\to\infty} \lambda(t) B(t) e^{-\rho t} = 0; \tag{A.8}$$

$$\lim_{t\to\infty} q(t) K(t) e^{-\rho t} = 0; \tag{A.9}$$

Admissibility conditions

$$B_0 = B(0), K_0 = K(0). \tag{A.10}$$

Appendix B. Non feasible steady states

Following the discussion in section 4, on the computation of the non feasible set of steady states, $\widetilde{Z}^* \in \Re$. We show in this section under what conditions this set of steady states can be described analytically or numerically. We start by defining the general expressions describing the intersection of two quadratic curves as,

$$\underbrace{\frac{rd}{2}(Z_2^*)^2}_{AX^2} - \underbrace{Z_2^* Z_4^*}_{BXY} + \underbrace{\frac{h}{2}(Z_4^*)^2}_{CY^2} + \underbrace{(r+\delta) Z_2^*}_{DX} + \underbrace{Z_4^*}_{EY} - \underbrace{r_k}_{F} = 0, \tag{B.1}$$

$$-\underbrace{\frac{rd}{2h}(Z_2^*)^2}_{GX^2} + \underbrace{rdZ_2^* Z_4^*}_{HXY} - \underbrace{\frac{1}{2}(Z_4^*)^2}_{IY^2} + \underbrace{\frac{rd}{h} Z_2^*}_{JX} + \underbrace{(r+\delta) Z_4^*}_{KY} + \underbrace{(r+\delta-r_k)\frac{1}{h}}_{L} = 0. \tag{B.2}$$

It is now convenient to characterize the geometry of the quadratic curves described by (B.1) and (B.2). We discard the case of infinitely many equilibrium and the hypothesis that these quadratic curves are represented by degenerate conic sections[48]. Following this assumption, we characterize each quadratic curve by determining the quantities:

$$B^2 - 4AC = 1 - rdh; \tag{B.3}$$

$$H^2 - 4GI = rd^2 - \frac{rd}{h} = rd\left(rd - \frac{1}{h}\right). \tag{B.4}$$

Following (B.3) the curve (B.1) is defined by: (i) a hyperbola when $hrd < 1$; (ii) a parabola when $hrd = 1$; and (iii) an ellipse when $hrd > 1$. Given (B.4) the curve (B.2) is: (i) a hyperbola when $hrd < 0 \wedge hrd > 1$; (ii) a parabola when $hrd = 1$; and (ii) and ellipse when $0 < rdh < 1$. The general solution to the system defined by (B.1) and (B.3) is given by a fourth order equation. We can solve this system analytically when the parabola constraint is considered, as it allows for a reduction of the fourth order equation to a second order one. To solve the intersection of (B.1) and (B.2), assuming $rdh = 1$, it is convenient to rearrange (B.1) in the following fashion[49],

$$Y^2 = -\frac{A}{C}X^2 - \frac{B}{C}XY - \frac{D}{C}X - \frac{E}{C}Y - \frac{F}{C}. \tag{B.5}$$

Substituting the Y^2 term in (B.2) by the expression given in (B.5) we obtain:

$$\left(G - \frac{AI}{C}\right)X^2 + \left(H - \frac{BI}{C}\right)XY + \left(J - \frac{DI}{C}\right)X + \left(K - \frac{EI}{C}\right)Y + L - \frac{FI}{C} = 0. \tag{B.6}$$

Recall now that the first two terms in (B.6) are given by the following expressions, $G - AIC^{-1} = -rd(2h)^{-1} + rd(2h)^{-1} = 0$ and $H - BIC^{-1} = rd - h^{-1}$. When $rdh = 1$, the second term vanishes and (B.6) reduces to,

$$X = -\frac{KC - EI}{JC - DI}Y - \frac{LC - FI}{JC - DI}. \tag{B.7}$$

Substituting now (B.7) in (B.1), the solution for the intersection of two parabolas is described by,

$$\begin{aligned} &\left[\tfrac{KC-EI}{JC-DI}\left(A\tfrac{KC-EI}{JC-DI} - B\right) + C\right]Y^2 + \left[2A\tfrac{(KC-EI)(LC-FI)}{(JC-DI)^2} - \tfrac{B(LC-FI)+D(KC-EI)}{JC-DI} + E\right]Y + \\ &+ \left[\tfrac{LC-FI}{JC-DI}\left(A\tfrac{LC-FI}{JC-DI} - D\right) + F\right] = 0. \end{aligned} \tag{B.8}$$

[48] Conic sections are degenerate when the determinant arising from the matrix representation of the curve is equal to zero.

[49] To allow for a clearer presentation, we shall use the general expressions of both quadratic curves throughout this section.

The non feasible steady states when $hrd = 1$ are given by

$$Z_2^* = -\frac{\left(r+\delta+\frac{1}{h}\right)h}{rd+r+\delta}Z_4^* - \frac{\delta+r-2r_k}{rd+r+\delta} \quad \text{and} \quad Z_4^* = \frac{-\Phi \pm \sqrt{\Phi^2-4\Theta\Psi}}{2\Theta}, \tag{B.9}$$

where Θ, Φ and Ψ are defined by the following set of expressions:

$$\begin{gathered}\Theta = \left[\tfrac{KC-EI}{JC-DI}\left(A\tfrac{KC-EI}{JC-DI}-B\right)+C\right], \quad \Phi = \left[2A\tfrac{(KC-EI)(LC-FI)}{(JC-DI)^2} - \tfrac{B(LC-FI)+D(KC-EI)}{JC-DI} + E\right] \\ \text{and} \quad \Psi = \left[\tfrac{LC-FI}{JC-DI}\left(A\tfrac{LC-FI}{JC-DI}-D\right)+F\right].\end{gathered} \tag{B.10}$$

Substituting Θ, Φ and Ψ by system parameters we obtain:

$$\Theta = 2h, \quad \Phi = \frac{2(r+\delta-2r_k)}{rd+r+\delta} - (r+\delta)h + 1 \quad \text{and} \quad \Psi = \frac{r+\delta-2r_k}{rd+r+\delta}\left[\frac{rd}{2}\left(\frac{r+\delta-2r_k}{rd+r+\delta}\right)-(r+\delta)\right] - r_k. \tag{B.11}$$

We now describe a numerical algorithm for the computation of the general solution of the system defined in (B.1) and (B.2). We propose a two step solution to solve this problem in a robust and efficient fashion. First, we define a solution that is linear in terms of one of the coordinates solution of (B.1) and (B.2). We choose coordinate Y for this purpose. Multiplying (B.1) by I and (B.2) by C, and imposing equality between the resulting expressions, we obtain the solution for Y in terms of the coordinate solution of X,

$$Y = \frac{(GC-AI)X^2 + (JC-DI)X + (LC-FI)}{(EI-KC)+(BI-HC)X}. \tag{B.12}$$

We now have to determine the coordinate solution for X. This solution is given by a fourth order polynomial. First we rearrange the original expressions, (B.1) and (B.2), as quadratic polynomials in terms of Y coordinate. Then we set the resulting system in a *Sylvester* matrix form:

$$\begin{bmatrix} C & BX+E & AX^2+DX+F & 0 \\ 0 & C & BX+E & AX^2+DX+F \\ I & HX+K & GX^2+JX+L & 0 \\ 0 & I & HX+K & GX^2+JX+L \end{bmatrix} \begin{bmatrix} Y^3 \\ Y^2 \\ Y^1 \\ Y^0 \end{bmatrix} = 0. \tag{B.13}$$

In order to obtain the coordinate solution in terms of X, we follow *Bezout's* theorem, and determine the resultant of the two original polynomials. To obtain the resultant, we set the determinant of the *Sylvester* matrix defined in (B.13) equal to zero. This condition is given by:

$$\begin{gathered}\left[C\left(GX^2+JX+L\right) - I\left(AX^2+DX+F\right)\right]^2 - \\ -\left[C(HX+K)-I(BX+E)\right]\left[(BX+E)\left(GX^2+JX+L\right)-(HX+K)\left(AX^2+DX+F\right)\right] = 0.\end{gathered} \tag{B.14}$$

Solving the above expression, we obtain the fourth order polynomial in terms of X coordinates. After a fair amount of calculus we obtain the following equation,

$$b_0X^4 + b_1X^3 + b_2X^2 + b_3X + b_4 = 0, \tag{B.15}$$

where the coefficients of this polynomial are given by the following expressions:

$$b_0 = (CG-IA)^2 + (HA-BG)(CH-IB); \tag{B.16}$$

$$b_1 = 2(CJ-ID)(CG-IA) - (KB+HE)(CG+IA) + (BJ-HD)(IB-CH) + 2(CHKA+IBEG) \ ; \tag{B.17}$$

$$\begin{gathered}b_2 = (CJ-ID)^2 + 2(CL-IF)(CG-IA) + (HF-BL)(CH-IB) - \\ -(KB+HE)(CJ+ID) + (KA-EG)(CK-IE) + 2(CHKD+IBEJ)\end{gathered} \ ; \tag{B.18}$$

$$b_3 = 2(CL-IF)(CJ-ID) - (KB+HE)(CL+IF) + (KD-EJ)(CK-IE) + 2(CHKF+IBEL) \ ; \tag{B.19}$$

$$b_4 = (CL-IF)^2 + (KF-EL)(CK-IE). \tag{B.20}$$

To finish this procedure, we must now employ a polynomial solver and obtain the coordinate solution in terms of X, given the solution defined in (B.15) to (B.20), and then substitute this solution in (B.12) to obtain the corresponding Y coordinate[50].

[50]For this purpose we built a C routine and compiled our code with the GNU scientific library(GSL) polynomial solver, which is based on the *Horner's* method for stability. We then obtain absolute computation errors by substituting the numerical solution in the original system, (B.1) and (B.2), and test their accuracy for an error tolerance defined by $|\max_{error}\{X^*, Y^*\}| \leq 10^{-5}$. For this error tolerance , all computed solutions consistent with, $X^*, Y^* \in \Re$, were accepted. We confirmed this procedure by running a routine in MATLAB using the built-in polynomial solver function *roots* and no significant differences were found.

Appendix C. Linearized dynamics and non-degeneracy conditions

Recall that given the restrictions described in section 4, the *Jacobian* in the vicinity of the economic meaningful steady states, $\widetilde{Z}^{**}$, is given generically by:

$$J^{**} = \begin{bmatrix} 0 & \frac{rd}{\gamma-1} J_{1,3}^{**} & J_{1,3}^{**} \\ 1 & J_{2,2}^{**} & J_{2,3}^{**} \\ 0 & \frac{rd}{h} J_{2,3}^{**} & J_{2,2}^{**} \end{bmatrix}. \tag{C.1}$$

The characteristic equation for this *Jacobian* comes,

$$(\Lambda^{**})^3 - 2J_{2,2}^{**}(\Lambda^{**})^2 - \Lambda^{**}\left(\frac{rd}{\gamma-1}J_{1,3}^{**} - \left(J_{2,2}^{**}\right)^2 + \frac{rd}{h}\left(J_{2,3}^{**}\right)^2\right) - rdJ_{1,3}^{**}\left(\frac{1}{h}J_{2,3}^{**} - \frac{1}{\gamma-1}J_{2,2}^{**}\right) = 0, \tag{C.2}$$

where Λ^{**} stands for the eigenvalues solving the characteristic polynomial in the vicinity of $\tilde{Z}_i^{**}$. The condition guaranteeing the *Jacobian* defined in (C.1) is non-degenerate, $Det\,(J^{**}) \neq 0$, is given by:

$$rdJ_{1,3}^{**}\left(\frac{J_{2,3}^{**}}{h} - \frac{J_{2,2}^{**}}{\gamma-1}\right) \neq 0 \Longrightarrow rdJ_{1,3}^{**} \neq 0 \wedge J_{2,3}^{**} \neq \frac{h}{\gamma-1}J_{2,2}^{**}. \tag{C.3}$$

We now focus on the linearized dynamics in the vicinity of of the non feasible set of steady states. Given the restrictions described in section 4 the *Jacobian* in the vicinity of $\widetilde{Z}^{*}$, is given by:

$$J^{*} = \begin{bmatrix} J_{1,1}^{*} & 0 & 0 \\ 1 & J_{2,2}^{*} & J_{2,3}^{*} \\ 0 & \frac{rd}{h} J_{2,3}^{*} & J_{2,2}^{*} \end{bmatrix}. \tag{C.4}$$

The characteristic equation for this *Jacobian* comes,

$$\left(J_{1,1}^{*} - \Lambda^{*}\right)\left[\left(J_{2,2}^{*} - \Lambda^{*}\right)^2 - \frac{rd}{h}\left(J_{2,3}^{*}\right)^2\right] = 0, \tag{C.5}$$

where Λ^{*} stands for the eigenvalues solving the characteristic polynomial in the vicinity of $\tilde{Z}_i^{**}$. Non-degeneracy condition, $Det\,(J^{*}) \neq 0$, impose the following restriction,

$$J_{1,1}^{*}\left[\left(J_{2,2}^{*}\right)^2 - \frac{rd}{h}\left(J_{2,3}^{*}\right)^2\right] \neq 0 \Longrightarrow J_{1,1}^{*} \neq 0 \wedge J_{2,2}^{*} \neq \pm J_{2,3}^{*}\sqrt{\frac{rd}{h}}. \tag{C.6}$$

The solution of the characteristic equation defined in (C.5) is given by:

$$\Lambda^{*} = J_{1,1}^{*} \wedge \Lambda^{*} = J_{2,2}^{*} \pm J_{2,3}^{*}\sqrt{\frac{rd}{h}}. \tag{C.7}$$

Appendix D. Local bifurcation analysis

In this section, we provide the analytical conditions for the existence of saddle-node and fold-hopf bifurcations, following the discussion in section 5. In order to put forward the sufficient conditions for existence of saddle-node bifurcations in this system, we first start by proving that the bifurcation constraint, r_k^{**}, given in (42), is consistent with $Det\,(J^{**}) = 0$. Recall that according to *Viète*'s theorem the product of eigenvalues is given by,

$$\prod_{j=1}^{3} \Lambda_i^{**} = rdJ_{1,3}^{**}\left(\frac{1}{h}J_{2,3}^{**} - \frac{1}{\gamma-1}J_{2,2}^{**}\right), \tag{D.1}$$

where j is the eigenvalue index. Since $rdJ_{1,3}^{**} \neq 0$, we require that the following condition is fulfilled,

$$\frac{1}{\gamma-1}\left(r + rdZ_2^{**} + \delta - Z_4^{**}\right) = \left(1 + hZ_4^{**} - Z_2^{**}\right)\frac{1}{h}. \tag{D.2}$$

Substituting the equilibrium expression for Z_2^{**}, (35), and solving in terms of Z_4^{**}, we confirm that r_k^{**} is consistent with the existence of this singularity, $\Lambda_1^{**} = 0$, and equal to the equilibrium expression for $Z_4^{**}\left(r_k^{**}\right)$, given in (43),

$$Z_4^{**} = \frac{-\left(\rho + \delta\gamma + \frac{(\gamma-1)(\rho-r+\delta(\gamma-1))}{hrd} - \frac{\gamma-1}{h}\right)}{-\frac{(\gamma-1)^2}{hrd} - 2\gamma + 1} = Z_4^{**}\left(r_k^{**}\right). \tag{D.3}$$

In three dimensional systems a saddle-node bifurcation occurs if the remaining eigenvalues are of opposite signs, $\Lambda_2^{**} \cdot \Lambda_3^{**} < 0$. Following (D.2), the characteristic equation is now given by:

$$-\Lambda^{**}\left[\left(J_{2,2}^{**} - \Lambda^{**}\right)^2 - \frac{rdh}{\gamma-1}\left(J_{2,2}^{**}\right)^2 - \frac{rd}{\gamma-1}J_{1,3}^{**}\right] = 0. \tag{D.4}$$

The eigenvalues at the bifurcation point are thus given by,

$$\Lambda_1^{**} = 0 \wedge \Lambda_{2,3}^{**} = J_{2,2} \pm \sqrt{\frac{rdh}{\gamma-1}\left(J_{2,2}^{**}\right)^2 + \frac{rd}{\gamma-1}J_{1,3}^{**}}. \tag{D.5}$$

The existence of a saddle-node bifurcation can be put in terms of parameter d. We have a saddle-node if

$$J_{1,3}^{**} < -h\left(J_{2,2}^{**}\right)^2 \text{ and } d > 0; \text{ or if } J_{1,3}^{**} > -h\left(J_{2,2}^{**}\right)^2 \text{ and } d < 0. \tag{D.6}$$

To define analytically the fold-hopf bifurcation point, it is convenient to continue equilibrium from the saddle node bifurcation defined by r_k^{**}. Continuing equilibrium from this point a fold-hopf bifurcation is guaranteed to exist if $J_{2,2}^{**}\left(r_k^{**}\right) = 0 \wedge J_{2,3}^{**}\left(r_k^{**}\right) = 0$, and provided that $d < 0$. At this singular point we have a zero eigenvalue, $\Lambda_1^{**} = 0$, and the remaining eigenvalues, $\Lambda_{2,3}^{**}$, are given by a pure imaginary conjugate pair, following the result in (D.5). The expression for the non negative eigenvalues is given by,

$$\Lambda_{2,3}^{**} = \pm\sqrt{\frac{rd}{\gamma-1}J_{1,3}^{**}}, \quad \frac{rd}{\gamma-1}J_{1,3}^{**} < 0. \tag{D.7}$$

To obtain the parameter constraint required for the existence of a codimension two fold-hopf bifurcation, we have to solve the system given by $J_{2,2}^{**}\left(r_k^{**}\right) = 0 \wedge J_{2,3}\left(r_k^{**}\right) = 0$. Substituting we obtain,

$$Z_4^{**}\left(r_k^{**}\right) = \frac{r+\delta+rd}{1-hrd} \wedge Z_2^{**}\left(r_k^{**}\right) = 1 + hZ_4^{**}\left(r_k^{**}\right). \tag{D.8}$$

Substituting (D.8) in (D.3), we obtain the second parameter condition, in terms of parameter ρ. This condition is given by the following expression,

$$\rho^{**} = -\left(\frac{hrd}{\gamma-1+hrd}\right)\left[\delta\gamma + \frac{(\gamma-1)(\delta(\gamma-1)-r)}{hrd} - \frac{\gamma-1}{h} - \left(\frac{r+\delta+rd}{1-hrd}\right)\left(-\frac{(\gamma-1)^2}{hrd} - 2\gamma + 1\right)\right]. \tag{D.9}$$

Appendix E. Geometric analysis of the quadric nullcline surfaces

To classify the nullclines described by the quadrics, $\dot{Z}_2(t), \dot{Z}_4(t) = 0$, it is first convenient to redefine these surfaces as a matrix product, $\chi^T\Sigma_i\chi^T$, where $\chi = [Z_2(t), Z_4(t), Z_1(t), 1]$. The matrix Σ_2 for the nullcline $\dot{Z}_2(t) = 0$ and its upper sub-matrix, $\Sigma_2^{'}$, are given in general terms, following the notation in Appendix B, by:

$$\Sigma_2 = \begin{bmatrix} A & \frac{B}{2} & 0 & \frac{D}{2} \\ \frac{B}{2} & C & 0 & \frac{E}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{D}{2} & \frac{E}{2} & \frac{1}{2} & F \end{bmatrix} \text{ and } \Sigma_2^{'} = \begin{bmatrix} A & \frac{B}{2} & 0 \\ \frac{B}{2} & C & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{E.1}$$

Analogously the matrix Σ_4 for the nullcline $\dot{Z}_4(t) = 0$ and its upper sub-matrix, $\Sigma_4^{'}$, are given by,

$$\Sigma_4 = \begin{bmatrix} G & \frac{H}{2} & 0 & \frac{J}{2} \\ \frac{H}{2} & I & 0 & \frac{K}{2} \\ 0 & 0 & 0 & 0 \\ \frac{J}{2} & \frac{K}{2} & 0 & L \end{bmatrix} \text{ and } \Sigma_4^{'} = \begin{bmatrix} G & \frac{H}{2} & 0 \\ \frac{H}{2} & I & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{E.2}$$

To define these geometric surfaces we need to define the following quantities: (i) $\det(\Sigma)$, $\det\left(\Sigma'\right)$; (ii) $\operatorname{rank}(\Sigma)$, $\operatorname{rank}\left(\Sigma'\right)$; and (iii) $\det\left(\Sigma' - \pi I\right) = 0$. Where π is the eigenvalue solution to the characteristic equation of Σ' and I the identity matrix. For the first quadric surface, $\dot{Z}_2(t) = 0$, we obtain the following quantities. When $hrd \neq 1$, $\det(\Sigma_2) = (1 - hrd)/16$ and the matrix has full rank, $\operatorname{rank}(\Sigma_2) = 4$, while $\det\left(\Sigma_2'\right) = 0$ and $\operatorname{rank}\left(\Sigma_2'\right) = 2$. When $hrd = 1$, $\det(\Sigma_2) = 0$ and $\operatorname{rank}(\Sigma_2) = 2$, while $\operatorname{rank}\left(\Sigma_2'\right) = 1$. The surface is a parabolic cylinder when $hrd = 1$. Now we need to evaluate the signs of the non negative eigenvalue solutions, π_2, for Σ_2'. These are given by,

$$\pi_2 = 0 \vee \pi_2 = \frac{-\frac{1}{2}(rd+h) \pm \sqrt{\frac{1}{4}(rd+h)^2 + 1 - rdh}}{-2}. \tag{E.3}$$

When $hrd > 1$, we have eigenvalues with the same signs. The quadric $\dot{Z}_2(t) = 0$ is: (i) an elliptic paraboloid when $hrd > 1$; and (ii) a hyperbolic paraboloid when $hrd < 1$. For the second quadric, $\dot{Z}_4(t) = 0$, we obtain the following quantities. When $hrd \neq 1$, $\det(\Sigma_4) = 0$ and $\operatorname{rank}(\Sigma_4) = 3$, while $\det\left(\Sigma_4'\right) = 0$ and $\operatorname{rank}\left(\Sigma_4'\right) = 2$. When $hrd = 1$,$\operatorname{rank}(\Sigma_4) = 3$ and $\operatorname{rank}\left(\Sigma_4'\right) = 1$, the surface is a parabolic cylinder. Now we need to evaluate the eigenvalues, π_4, of Σ_4'. These are given by the following expressions,

$$\pi_4 = 0 \vee \pi_4 = \frac{\frac{1}{2}\left(\frac{rd}{h} + 1\right) \pm \sqrt{\frac{1}{4}\left(\frac{rd}{h} + 1\right)^2 + rd\left(rd - \frac{1}{h}\right)}}{-2}. \tag{E.4}$$

When $0 < rdh < 1$, the quadric, $\dot{Z}_4(t) = 0$, is an elliptic cylinder[51]. When $hrd < 0 \wedge hrd > 1$, the quadric, $\dot{Z}_4(t) = 0$, is a hyperbolic cylinder.

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