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Portfolio Selection – A Technical Note†

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ABSTRACT

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This note develops the solutions of the static portfolio optimization problem in explicit matrix form. Three cases are contemplated and connected, with the derivation of relevant corner solutions: the unconstrained problem in the presence of risky assets only, the constrained one, and the presence of a risk-free asset. The use of a generalized form for the budget constraint allows us to use the structure to study the behavior of a complete borrower – subject or not to liquidity constraints – and infer the price of pure risk.

Some properties of the several solutions are highlighted. The rationale for a linear relation between the standard deviation and the expected return of the unitary application in an efficient portfolio is derived. Requirements for useful existence in the market of any given security are established. Additionally, we infer the expected co-movement properties of efficient and the global market – or any other – portfolio.

JEL Classification: G11; G12; G24. C61. D81.

Keywords: Portfolio choice; Mean-variance; CAPM; Quadratic programming; Price of risk.

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Portfolio Selection – A Technical Note

"the weight of gold for all golden vessels for each service, the weight of silver vessels for each service, the weight of the golden lampstands and their lamps, the weight of gold for each lampstand and its lamps, the weight of silver for a lampstand and its lamps, according to the use of each lampstand in the service". In *1 Chronicles*, 28: 14-15.

1. Introduction. Usually, the finance literature [1] does not provide an explicit matrix form solution for the mean-variance efficient portfolio associated to the Markowitz (1952 and 1959) and Tobin's (1958) model, or its equilibrium version Capital Asset Pricing Model [2]. The exceptions – as Campbell, Lo and MacKinlay (1997) - do not explore the corner solutions. It is the purpose of this note to circumvent such shortcoming: unlike previous literature, we rely on a generalized budget constraint, which allows a clearer study of the implications of introducing borrowing, in particular, full borrowing. Along the text, we note properties of the optimal solutions and connect the several results, stating the relevant market line boundaries for their applicability.

We start by the unconstrained problem – stated in section 2 in its primal and dual forms -, where we also explore some properties of un-held securities. In section 3 we introduce the risk-free asset and the budget constraint; borrowing at the risk-free rate naturally arises a special case. The constrained borrowing setting is then solved for in section 4. Section 5 exploits the generalized budget constraint in order to represent full fund borrowing – which allows us to derive a measure of the price of pure risk. Section 6 concludes interpreting the market betas.

2. Unrestricted Funds. Admit that in the market there are n assets. Each of them, i, is present in the market in such a way that u_i denotes its *aggregate* yield in the period – u denotes the corresponding column vector and μ its expected value. The covariance matrix of those n aggregate returns is known, a symmetric positive-definite matrix V, with generic element σ_{ij}. The n returns - corresponding to n different assets - or any linear combination of them, are never perfectly correlated: they belong to the maximum number of existing assets with which a non-singular matrix V can be constructed.

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[1] In particular, textbooks: Brealey and Myers (2003), Cuthbertson (1996), Wilmott (2001). Nor more theoretical and uncertainty oriented: Hirshleifer and Riley (1992), Laffont (1989).

[2] See Sharpe (1964), Lintner (1965) and Black (1972) among others. Duffie (1991) presents a recent version and overview of similar material.

An efficient portfolio is going to be considered one that exhibits minimal variance for given expected return. Denote by W_i the fraction of (total existing in the market) equity i contained in a given portfolio and W the corresponding column-vector. Of course, the "market portfolio" was defined in such a way that $W^M = L$, where L denotes a column vector of 1's: L' = [1 1 … 1]'. Consider that a (minimal) mean return of α is sought – related to the global amount an investor wants to apply - and the objective is to diversify the portfolio composition – choosing a suitable vector W, W_α^* - in such a way as to minimize the global variance. That is:

$$(1) \qquad \underset{W}{Min} \quad W' V W$$

$$\text{s.t.} \quad \mu' W \geq \alpha$$

$$W_i \geq 0 , W_i \leq 1$$

The previous problem is a conventional quadratic programming one [3], and the optimal solution should obey Kuhn-Tucker conditions if corner solutions are to be binding [4]. We will use the principles behind both to derive an explicit form for as well as properties of the optimal portfolio weights, W_α^*.

Let us assume that non-negativity constraints are satisfied in the optimal solution – all assets are required to insure minimum variance; the constraint is going to be met in equality and does not exhaust the market. Then, the optimal solution will minimize the Lagrangean:

$$\underset{W,\lambda}{Min} \; L(W, \lambda) = W' V W + \lambda (\alpha - \mu' W)$$

where λ denotes the multiplier that must be non-negative. F.O.C. imply [5]:

$$(2) \qquad \frac{\partial L}{\partial W} = 2 W' V - \lambda \mu' = 0 \text{ (a (1 x n) vector)}$$

$$(3) \qquad \frac{\partial L}{\partial \lambda} = \alpha - \mu' W = 0 \text{ (a scalar)}$$

Transposing the first condition and solving for W, we derive:

$$(4) \qquad W = \frac{1}{2} \lambda V^{-1} \mu$$

[3] See Taha (1982), p. 776-780, for example. Also, problem 4-Q, p. 67, of Intriligator (1971), referring Markowitz (1959). The objective function is convex, the restrictions are linear so S.O.C. are satisfied.

[4] See Taha (1982), p. 753-757, for example.

As $\mu' W = \frac{1}{2} \lambda \mu' V^{-1} \mu = \alpha$ for the constraint to be satisfied, the multiplier λ, representing how the optimized minimand (the minimum variance) responds to (a unitary increase of) the required earnings α in the optimal solution:

$$\lambda_\alpha^* = 2 (\mu' V^{-1} \mu)^{-1} \alpha = \frac{\partial(W_\alpha^{*}{}' V W_\alpha^*)}{\partial \alpha} \quad (5)$$

Replacing in (4):

$$W_\alpha^* = V^{-1} \mu (\mu' V^{-1} \mu)^{-1} \alpha \quad (6)$$

W_α^* is proportional to α, the required return. The relative weights of the n assets in an optimal portfolio are mean (α) invariant [6]:

$$\frac{W_\alpha^*}{L' W_\alpha^*} = \frac{V^{-1}\mu}{L' V^{-1}\mu} \quad (7)$$

$(\mu' V^{-1} \mu)$ is positive, once V is positive-definite; hence, as the sum of (and each of…) the controls must be positive, $L' W_\alpha^* > 0$, - using (6) - $L' V^{-1} \mu$ is also positive.

The minimum variance of the optimal portfolio is then:

$$\sigma_\alpha^{*2} = W_\alpha^{*}{}' V W_\alpha^* = (\mu' V^{-1} \mu)^{-1} \alpha^2 \quad (8)$$

Its standard deviation is going to be *proportional* to the expected value of required return, α [7]:

$$\sigma_\alpha^* = (\mu' V^{-1} \mu)^{-1/2} \alpha \quad (9)$$

Notice that:

$$\lambda_\alpha^* = 2 \frac{\sigma_\alpha^{*2}}{\alpha} \quad (10)$$

To increase the expected return of an efficient portfolio by one unit, the optimal portfolio will see its variance proportionally increased by (: using (8), $\frac{d\sigma_\alpha^{*2}}{d\alpha} \frac{1}{\sigma_\alpha^{*2}} = 2 (\mu' V^{-1} \mu)^{-1} \alpha \frac{1}{\sigma_\alpha^{*2}}$; replacing (8), we obtain) $\frac{2}{\alpha}$.

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[5] We follow Dhrymes (1978) conventions with respect to notation in matrix differentiation. The generalized interior solution for the quadratic programme is depicted in Appendix 1; it can be also be developed from the matrix partition form of the system - see Lange (2004).

[6] These weights would have a more adequate meaning if the problem is stated so that μ and V measure mean and covariances of monetary units of the n assets. See below and footnote 10.

. As is well-known, the results could have been derived from the (in economics jargon) dual problem – maximization of the expected return with portfolio variance not exceeding a given threshold, call it σ_α^{*2}:

$$\underset{W}{Max} \quad \mu' W$$
$$\text{s.t.} \quad W' V W \leq \sigma_\alpha^{*2}$$
$$W_i \geq 0 , W_i \leq 1$$

with lagrangean: $\underset{W,\gamma}{Max} \ L(W, \gamma) = \mu' W + \gamma (\sigma_\alpha^{*2} - W' V W)$

It is straight-forward to derive that the optimal W_σ^* will be those of (6) with α replaced by (9): as a function of the required minimum variance,

(11) $$W_\sigma^* = V^{-1} \mu \ (\mu' V^{-1} \mu)^{-1/2} \sigma_\alpha^*$$

At a value of σ_α^* obeying (9), $W_\sigma^*{}'\mu = \alpha$, and the two solutions, W_α^* and W_σ^*, coincide. Also,

(12) $$\gamma^* = \frac{1}{2\,\sigma_\alpha^{*2}} (\mu' V^{-1} \mu)^{1/2} = \frac{\partial W_\sigma^*{}'\mu}{\partial \sigma_\alpha^2}$$

and equals $\frac{1}{\lambda_\alpha^*}$ of (5) if we replace α according to (9). We will keep form (1) in the several sections.

. If some of the controls in W_α^* derived as above are negative, we can always replicate the solution discarding them one at a time out of the basket W and retry [8]. Eventually, we will attain the securities of the internal solution, which must be mixed as the n above.

In the optimal solution in which asset r reaches the corner, the r-th equation in system (2) will hold in inequality in such a way that: $2\ W^{\#*}{}'\ V^{\#}_r > \lambda^*\ \mu_r$, where μ_r is the expected return of r, $W^{\#*}$ is the optimal portfolio vector, enlarged to include $W_r^* = 0$, and $V^{\#}_r$ denotes the r-th column of an enlarged "V" matrix, also referring the "idle" asset r. $W^{\#*}{}'\ V^{\#}_r = W_\alpha^*{}'\ V_r = V_r'\ W_\alpha^*{}^*$ where V_r contains the covariances of r with all other n assets. Then, for an idle asset – bound to disappear, once no one would hold it; or investment projects that will not be carried out:

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[7] See Tobin (1958), p. 84, expression (3.25).

[8] See Taha (1982), p. 751, for a stepwise recommendation to approach the optimal solution of generic problems of the same kind.

(13) $\quad 2\, W_\alpha^{*}{}'\, V_r > \lambda_\alpha^{*}\, \mu_r \quad$ or $\quad \mu_r < V_r'\, V^{-1}\, \mu$

As a corollary, one concludes that for an asset r not [9] in the set of n assets that enter V to survive in the market

(14) $$\mu_r = V_r'\, V^{-1}\, \mu$$

That will be required if it is interchangeable with one, or replicated by a linear combination of the n assets of the efficient portfolio. The expression states the mean of asset r, $E[u_r] = \mu_r$ as a linear combination of the means of other active assets in the efficient portfolio; then, if also $u_r = V_r'\, V^{-1}\, u$, for an existing security not in the efficient portfolio, (14) holds and:

(15) $$\mathrm{Var}(u_r) = V_r'\, V^{-1}\, V_r$$

. The math would fall through if μ and V referred mean vector and variance-covariance matrix of unitary money units of the n market assets [10]. In such context, no assumptions are made with respect to the required budget, L' W - funds are assumed unlimited, or L' W_α^* is the required investment to attain return α with minimum variance – per unit of α, the required investment is

$$L'\, W_\alpha^* / \alpha = (L'\, V^{-1}\, \mu) / (\mu'\, V^{-1}\, \mu)$$

If there are limited funds, g, provided L' $W_\alpha^* < g$, i.e., using (6), if the required return per unit of investment, α' is smaller than $\alpha'^{\#}$:

(16) $$\alpha / g = \alpha' < \alpha'^{\#} = (\mu'\, V^{-1}\, \mu) / (L'\, V^{-1}\, \mu) = \frac{\mu' W_\alpha^*}{L' W_\alpha^*}$$

the previous solution holds. If not, a constrained solution must be sought. $\alpha'^{\#}$ is, of course equal to - using (7), $\frac{\mu' W_\alpha^*}{L' W_\alpha^*}$ - the expected return of the unit of investment in the optimal "risky assets basket" – the shares of which, by (7), are independent of α or g. Its standard-deviation – replacing (16) in (9) – is also budget and mean invariant:

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[9] Of course, (14) and (15) hold for any asset r included in the efficient portfolio, in which case V_r is the column of V containing r's variance and covariances with all other assets. That is, due to symmetry of V, $\mu = V'\, V^{-1}\, \mu$ and $V = V'\, V^{-1}\, V$ always.

(17) $$\sigma^*_{\alpha'^{\#}} = \sigma^*_{\frac{\mu' W^*_\alpha}{L' W^*_\alpha}} = (\mu' V^{-1} \mu)^{1/2} / (L' V^{-1} \mu)$$

Finally, the appropriate concept to use for μ is that of a rate of return (including appreciation) "net" of invested capital (it is r, not 1 + r…), which (but) we are implicitly assuming that will be recovered. That is, we want to maximize wealth after the realization of returns. If g is available, and we invest L' W, we will obtain u' W + L' W, but we still "kept" aside g – L' W (say, as cash-balances, as in Tobin's liquidity preference theory - then, these unused funds are indeed kept as an *existing risk-less asset* with zero return… Or used them for consumption.) that was not used [11]: summing both, we get W' u + g, which justifies optimization on W' u – and α is defined in terms of expected earnings excluding invested capital: the constraint is, still, W' μ + g ≥ α + g. Putting it differently, we get W' μ + L' W, but we had to "advance, put down" *our* (work, consumption option…) L'W before-hand…

3. Risk-Free Asset and the Budget Constraint. Let there be a risk-free asset in the economy and W_0 denote the amount to be invested in such an asset, with unit return μ_0, and a potential choice along with the other n assets. Such risk-free asset would be the only one held: under the conditions of the previous problem, if $\mu_0 > 0$, it would always be possible to achieve a given α with zero variance by the appropriate choice of $W_0 = \alpha/\mu_0$ (in the dual perspective, to attain a given variance we could always improve expected return). Then, additional restrictions must be imposed (exist): in the previous problem, the only cost of expected return is risk, which with a risk-free (and free…) asset is totally neutralized...

Let g be the amount to be invested, μ_i refer the expected returns of a monetary unit applied to asset i, W_i the amount to allocate to it, V the covariance matrix of the returns to unitary applications in the n assets. Vector W and the scalar W_0 should satisfy:

(18) $$L' W + W_0 \leq g$$

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[10] When the available assets are rescaled, their return vector is pre-multiplied by a diagonal non-singular matrix P and (5), (8), (9) and (10) will stand. The new optimal portfolio will be $P^{-1} W^*_\alpha$. Relative weights – (7) – change – they are now measured in (different, meaningfully summable) monetary units of investment…-, to $\frac{P^{-1}V^{-1}\mu}{L' P^{-1}V^{-1}\mu}$, but they still are independent of α.

[11] And did (do) not get wasted… If they did, we should then replace μ by (μ + L) in the formulas and consider α as return plus investment – or replace it by α + g.

In any optimal solution, provided that $\mu_0 > 0$, this budget constraint will always be exhausted – i.e., work in equality. That because – recall the dual problem structure – if a minimum variance is achieved with less than g, we will always raise the expected value of the portfolio by allocating the rest of the budget to any totally safe security that is positively rewarded. This implies that we can always collapse it into the other restriction, stating the minimum required return:

$$\mu' W + \mu_0 W_0 = \mu' W + \mu_0 (g - L' W) \geq \alpha$$

Normalizing in (dividing the equation by) g is then justified – the implicit treatment of the problem in the shares (1 = L' W) is therefore commonly pursued in the literature -, but does not allow for g = 0. We shall keep the general format and both restrictions separated.

Again, μ refer earnings rates. We are maximizing the expected value of detained wealth, g, through (after) the investment decision: the returns plus invested capital ($u' W + \mu_0 W_0 + L' W + W_0$), plus the value of hypothetically unused resources that were kept aside ($g - L' W - W_0$) [12]. Summing both, we get $\mu' W + \mu_0 W_0 + g$: the implicit return constraint is $\mu' W + \mu_0 W_0 + g \geq \alpha + g$. As the budget constraint is going to be satisfied, it could also be stated in terms of W only as $u' W + \mu_0 (g - L' W) \geq \alpha$.

The appropriate lagrangean can then be written as:

$$\underset{W, W_0, \lambda, \nu}{Min} \quad W' V W + \lambda (\alpha - \mu' W - \mu_0 W_0) + \nu (g - L' W - W_0)$$

where λ and ν denote the multipliers (ν is stated as to be negative). F.O.C. imply:

(19) $$\frac{\partial L}{\partial W} = 2 W' V - \lambda \mu' - \nu L' = 0 \text{ (a (1 x n) vector)}$$

(20) $$\frac{\partial L}{\partial W_0} = -\lambda \mu_0 - \nu = 0 \text{ (a scalar)}$$

(21) $$\frac{\partial L}{\partial \lambda} = \alpha - \mu' W - \mu_0 W_0 = 0 \text{ (a scalar)}$$

(22) $$\frac{\partial L}{\partial \nu} = g - L' W - W_0 = 0 \text{ (a scalar)}$$

Replacing the second condition,

(23) $$\nu = -\lambda \mu_0$$

in the first condition and solving for W, we derive:

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[12] We could assume - superimpose - that these would have been nonetheless "lent", capitalized, at rate μ_0, and add $(g - L' W - W_0)(1 + \mu_0)$ – and sum with applied funds $u' W + \mu_0 (g - L' W) + g$. As long as $\mu_0 \geq 0$, either the budget constraint is satisfied and the term is zero, or it is indifferent.

(24) $$W = \frac{1}{2} \lambda\ V^{-1} (\mu - \mu_0 L)$$

Replacing the last FOC condition,

(25) $$W_0 = g - L' W$$

in the third:

(26) $$(\mu - \mu_0 L)' W = (\alpha - \mu_0 g)$$

As $(\mu - \mu_0 L)' W = \frac{1}{2} \lambda (\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L) = (\alpha - \mu_0 g)$, in the optimal solution $\lambda = \lambda_\alpha^*$:

(27) $$\lambda_\alpha^* = 2\ [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g) = \frac{\partial (W_\alpha^{*}{}' V W_\alpha^*)}{\partial \alpha}$$

Of course, $\alpha > \mu_0 g$: if not, zero variance would be achieved by just subscribing to the risk-less asset. Replacing in (24):

(28) $$W_\alpha^* = V^{-1} (\mu - \mu_0 L)\ [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g)$$

Among themselves, the relative weights of the n assets in the optimal portfolio are mean (α) *and* budget (g) invariant – defining the composition of (a unit of investment in) the optimal *risky basket* -, but *will depend on the expected return of the risk-less asset*, μ_0:

(29) $$\frac{W_\alpha^*}{L' W_\alpha^*} = \frac{V^{-1}(\mu - \mu_0 L)}{L' V^{-1}(\mu - \mu_0 L)}$$

As V is positive definite, $(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L) > 0$; and $L' W_\alpha^* = g > 0$, $L' V^{-1} (\mu - \mu_0 L) > 0$ and hence $L' V^{-1} \mu > \mu_0 L' V^{-1} L$. $[(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)] > 0$ thus implies (with $\mu_0 > 0$) $\mu' V^{-1} (\mu - \mu_0 L) > \mu_0 L' V^{-1} (\mu - \mu_0 L) > 0$: $\mu' V^{-1} \mu > \mu_0 \mu' V^{-1} L = \mu_0 L' V^{-1} \mu$. Then, it must be the case that:

(30) $$\mu' V^{-1} \mu > \mu_0 \mu' V^{-1} L = \mu_0 L' V^{-1} \mu > \mu_0^2 L' V^{-1} L > 0$$

For interior solutions,

(31) $$\mu_0 < \mu' V^{-1} \mu / (\mu' V^{-1} L) = \alpha'^{\#}$$

and

(32) $$\mu_0 < L' V^{-1} \mu / (L' V^{-1} L) = \alpha'^{\#\#\#}$$

otherwise, the asset composition must (have…) changed.

Per unit of investment, g:

(33) $$\frac{W_\alpha^*}{g} = V^{-1} (\mu - \mu_0 L)\ [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} \left(\frac{\alpha}{g} - \mu_0\right)$$

The weights of the risky assets in the optimal portfolio decrease with g to achieve a given α, and increase linearly with the required α at given g - and with the required α per unit of budget application, i.e., with $\alpha' = \frac{\alpha}{g}$. We could re-write the problem in terms of the same parameters, replacing α by $\alpha = g\,\alpha'$ in all the expressions; then, for given α', those weights (each asset share on the budget) do not depend on the budget g.

Complementarily:

(34) $W_0^* = g - L' V^{-1} (\mu - \mu_0 L) \; [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g)$

and

(35) $\frac{W_0^*}{g} = 1 - L' V^{-1} (\mu - \mu_0 L) \; [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\frac{\alpha}{g} - \mu_0)$

The optimal weight (share) of the risk-free asset in the budget decreases linearly with $\alpha' = \frac{\alpha}{g}$ and it is proportional to the desired surplus (excess) relative to the risk-free asset return, $(\alpha' - \mu_0)$ and it does not depend (otherwise) on g, the budget.

$W_0^* \geq 0$ and the solution defines the position of a lender at the risk-less asset rate iff – using (34):

(36) $$\frac{\alpha}{g} = \alpha' \leq [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L) + L' V^{-1} (\mu - \mu_0 L)\, \mu_0] /$$

$$/ [L' V^{-1} (\mu - \mu_0 L)] = \alpha'^{\#\#} =$$

$$= [\mu' V^{-1} (\mu - \mu_0 L)] / [L' V^{-1} (\mu - \mu_0 L)] = \frac{\mu' W_\alpha^*}{L' W_\alpha^*}$$

If W_0^* is smaller than 0, the solution represents that of a borrower at the risk-less-rate and it is only possible if, in fact, W_0 is unrestricted: if not, for $\alpha' > \alpha'^{\#\#}$ a restricted solution must be sought. $\alpha'^{\#\#}$ - using (29) - also equals the expected return of the (per) unit of investment in the (generic, due to (29)) optimal risky basket, $\frac{\mu' W_\alpha^*}{L' W_\alpha^*}$.

The minimum variance of the optimal portfolio is:

(37) $$\sigma_\alpha^{*\,2} = W_\alpha^{*}{}' \, V \, W_\alpha^* = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g)^2 =$$

$$= g^2 \; [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\frac{\alpha}{g} - \mu_0)^2$$

Its standard deviation is going to be *proportional* to the expected value of applied resources, g, if we redefine the target mean per unit of application, i.e., $\frac{\alpha}{g}$, and – at given g - still linear in α (or α'), now proportional to the excess return relative to the risk-free asset, $(\frac{\alpha}{g} - \mu_0) = (\alpha' - \mu_0)$.

(38) σ^*_α = g [(μ - μ_0 L)' V^{-1} (μ - μ_0 L)]$^{-1/2}$ ($\frac{\alpha}{g}$ - μ_0)

or $\sigma^*_{\alpha'}$ = $\frac{1}{g}$ σ^*_α = [(μ - μ_0 L)' V^{-1} (μ - μ_0 L)]$^{-1/2}$ (α'- μ_0)

The standard deviation of the unit application in the risky basket - σ^*_α / L'W* - is thus:

(39) $$\sigma^*_{\alpha^{\#\#}} = \sigma^*_{\frac{\mu' W^*_\alpha}{L' W^*_\alpha}} = \frac{[(\mu-\mu_0 L)'V^{-1}(\mu-\mu_0 L)]^{1/2}}{L'V^{-1}(\mu-\mu_0 L)}$$

It is now the case that:

(40) $$\lambda^*_\alpha = 2\,\frac{\sigma^{*2}_\alpha}{\alpha-\mu_0 g}$$

To increase the mean return of an efficient portfolio by one unit, the optimal portfolio will see its variance proportionally increased by $\frac{2}{\alpha-\mu_0 g}$ $\left(=\frac{d\sigma^{*2}_\alpha}{d\alpha}\frac{1}{\sigma^{*2}_\alpha}\right.$ by (37)). To increase the mean return per unit application on the efficient portfolio by one unit, the variance of such application is raised proportionately by $\frac{2}{\frac{\alpha}{g}-\mu_0} = \frac{2}{\alpha'-\mu_0}$.

Also, the change in variance achieved, for given α, when the budget increases by one unit is (negative and):

(41) $$\nu^*_\alpha = \frac{\partial(W^{*}_\alpha{}'VW^*_\alpha)}{\partial g} = -2\ \mu_0\ [(\mu - \mu_0 L)'\ V^{-1}\ (\mu - \mu_0 L)]^{-1}\ (\alpha - \mu_0 g) =$$

$$= -\ 2\,\mu_0\ \frac{\sigma^{*2}_\alpha}{\alpha-\mu_0 g}$$

As a final appraisal, we conclude that all the efficient portfolio formulas derived for the case where the risk-free asset was absent apply now if we replace the vector of mean returns μ by its deviation relative to the risk-free asset return, (μ - L μ_0) – the covariance matrix of those excess returns relative to μ_0 being, of course, also V – and if we replace α by (α - g μ_0) for the general efficient portfolio, by (α' - μ_0) for the unitary portfolio. That is, if we redefine the relevant returns (of a unitary portfolio) as the *deviation from the risk-free asset return*.

. Consider the possibilities offered for the *unitary* application, that is define α' = α / g or admit g = 1. In space ($\sigma^*_{\alpha'}$, α'), the opportunity set, bordered by (38) is a straight-line, crosses the vertical axis at level μ_0, and has slope [(μ - μ_0 L)' V^{-1} (μ - μ_0 L)]$^{1/2}$ –

translating how much more expected return can be obtained if one is willing to take an additional unit of standard-deviation in a portfolio -, which (because V is positive-definite) decreases with μ_0. It is a line like μ_0a, depicted in Fig. 1 below:

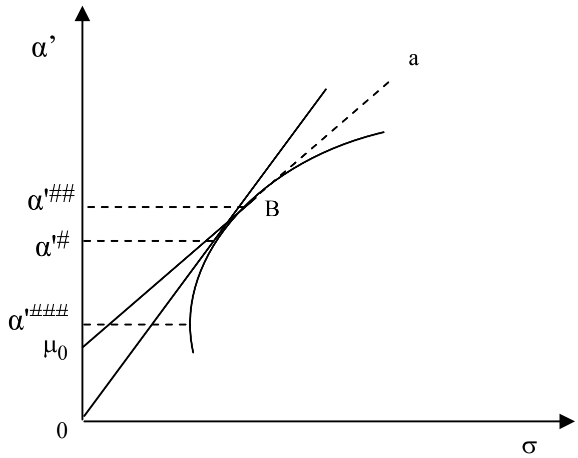

Fig. 1

For $\alpha' < \alpha'^{\#\#}$ - over μ_0B - the (risk-averse) individuals will combine – according to their preferences and budget – risk-less asset and optimal risky basket (of unit composition defined by (29)). For $\alpha' > \alpha'^{\#}$, the individuals borrow at the risk-less asset rate to finance investment in the risky basket.

. An un-held security r will be such that for the optimal solution the corresponding single equation in (19) will hold in inequality, i.e.:

(42) $\quad 2\, W_\alpha^{*}{}'\, V_r > \lambda_\alpha^{*}\, \mu_r + \nu_\alpha^{*} \quad$ or $\quad \mu_r < \mu_0 + V_r'\, V^{-1} (\mu - \mu_0\, L)$

where μ_r is the expected value of each of its units and V_r the column-vector containing the covariance between its unitary return and those of the included assets.

4. Borrowing Restrictions. We impose $W_i \geq 0$ for i =1,2,…,n: we are modeling demand over existing assets. We allow W_0* to be negative - the individual can lend W_0 as borrow $-W_0$ of the risk-free asset 0 at unit cost μ_0. For an institutional investor (say, bank, government) buying/promoting the portfolio, a negative W_0 may be possible and it has a meaning: it can take a “short position” on the risk-less asset, say borrowing at a fixed interest rate. Moreover, any loan has some collateral as long as it is applied in the purchase of assets – the assets themselves. Then the market line is indeed the straight line μ_0a.

If it must be the case that $W_0^* \geq 0$, then, for $\alpha' > \alpha'^{\#\#}$, we enter into the corner solution for which $W_0^* = 0$. Then the problem solves the lagrangean form:

$$\underset{W,W_0,\lambda,\nu}{Min} \quad W' V W + \lambda (\alpha - \mu' W) + \nu (g - L' W)$$

The optimal solution – provided the same assets keep being active in the portfolio - satisfies the system:

(43) $$2 V W = \lambda \mu + \nu L$$

(44) $$\alpha = \mu' W$$

(45) $$g = L' W$$

From the first expression,

(46) $$W = \frac{1}{2} V^{-1} (\lambda \mu + \nu L)$$

Pre-multiplying by μ' and L' respectively:

(47) $$\alpha = \mu' W = \frac{1}{2} \mu' V^{-1} (\lambda \mu + \nu L)$$

and:

(48) $$g = L' W = \frac{1}{2} L' V^{-1} (\lambda \mu + \nu L)$$

We can solve for:

(49) $$\nu^* = 2 [g \mu' V^{-1} \mu - \alpha L' V^{-1} \mu] / [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2] = $$
$$= 2 g [\mu' V^{-1} \mu - \alpha' L' V^{-1} \mu] / [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2] < 0$$

(50) $$\lambda^* = 2 [\alpha L' V^{-1} L - g L' V^{-1} \mu] / [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2] =$$
$$= 2 g [\alpha' L' V^{-1} L - L' V^{-1} \mu] / [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2] > 0$$

(51) $$W^* = V^{-1} \{\alpha [(L'V^{-1}L) \mu - g (L'V^{-1}\mu) L] + g [(\mu'V^{-1}\mu) L - (L'V^{-1}\mu) \mu]\} /$$
$$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2] =$$
$$= g V^{-1} \{\alpha' [(L' V^{-1} L) \mu - (L' V^{-1} \mu) L] + (\mu' V^{-1} \mu) L - (L' V^{-1} \mu) \mu\} /$$
$$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2]$$

The optimal asset shares, W* / L'W* = W* / g, are invariant to g, but now not to level of required expected return, α, and the homogeneity of the "optimal risky basket" breaks down. The unit return of optimal investments in risky assets equals that of the portfolio itself, is, of course, variable with, and equal to α': μ W* / L'W α / g = α'.

Inferring the opportunity locus:

(52) $$W^{*\prime} V W^* = \sigma_\alpha^{*2} =$$
$$[\alpha (\alpha L' V^{-1} L - g L' V^{-1} \mu) + g (g \mu' V^{-1} \mu - \alpha L' V^{-1} \mu)] /$$
$$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2]$$
$$= g^2 [\alpha' (\alpha' L' V^{-1} L - L' V^{-1} \mu) + (\mu' V^{-1} \mu - \alpha' L' V^{-1} \mu)] /$$
$$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2]$$

or

(53) $\sigma^*_{\alpha'} = \{[\alpha' (\alpha' L' V^{-1} L - L' V^{-1} \mu) + (\mu' V^{-1} \mu - \alpha' L' V^{-1} \mu)] /$
$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2]\}^{1/2}$

Now, (53) coincides with the standard-deviation of one unit application in risky assets and depends on the required α'.

Of course, at $\alpha' = \alpha'^{\#\#}$ - point B in Fig. 1 - the curve (53) touches the unrestricted linear line (38) – and must be to its south-east in space ($\sigma_{\alpha'}$, α'). Above it – for $\alpha' > \alpha'^{\#\#}$ - , the restricted curve would be below that line, slope upward, and be concave in space ($\sigma_{\alpha'}$, α'). That is:

(54) $\frac{d\sigma^*_{\alpha'}}{d\alpha'} = (\alpha' L' V^{-1} L - L' V^{-1} \mu)$
$[\alpha' (\alpha' L' V^{-1} L - L' V^{-1} \mu) + (\mu' V^{-1} \mu - \alpha' L' V^{-1} \mu)]^{-1/2} /$
$/ [L' V^{-1} L \mu' V^{-1} \mu - (L' V^{-1} \mu)^2]^{1/2}$

(55) $\frac{d^2\sigma^*_{\alpha'}}{d\alpha'^2} = [\alpha' (\alpha' L' V^{-1} L - L' V^{-1} \mu) + (\mu' V^{-1} \mu - \alpha' L' V^{-1} \mu)]^{-3/2}$

$\frac{d^2\sigma^*_{\alpha'}}{d\alpha'^2}$ must be positive. Hence, the new line – positively sloped for $\alpha' > \alpha'^{\#\#\#}$, that coincides with the bound (32), such that:

(56) $\alpha'^{\#\#\#} = (L' V^{-1} \mu) / (L' V^{-1} L)$

– must be concave in space ($\sigma_{\alpha'}$, α'). One can show easily show that it is tangent to the linear market line at $\alpha' = \alpha'^{\#\#} = [\mu' V^{-1} (\mu - \mu_0 L)] / [L' V^{-1} (\mu - \mu_0 L)]$ (of (36)) – that is, that (54) evaluated at $\alpha'^{\#\#}$ equals the slope of line (38) and that (53) at $\alpha'^{\#\#}$ equals the standard deviation given by (38) for the same level – it is drawn in Fig. 1.

The standard deviation reaches a minimum – also in Fig. 1 - at $\frac{d\sigma^*_{\alpha'}}{d\alpha'} = 0$, i.e., at $\alpha'^{\#\#\#}$, which for an interior solution – from (32) -, must be larger than μ_0.

. We should note the following: we could think that a specialization in risky assets would provide an opportunity given by the line $\sigma^*_{\alpha'} = (\mu' V^{-1} \mu)^{-1/2} \alpha'$. If $\mu_0 < (L' V^{-1} \mu) / (L' V^{-1} L)$ or $L' V^{-1} (\mu - \mu_0 L) > 0$, such line has higher slope than (38) in space ($\sigma^*_{\alpha'}$, α'), $\sigma^*_{\alpha'} = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1/2} (\alpha' - \mu_0)$, and crosses it at:

(57) $\alpha'^* = \mu_0 (\mu' V^{-1} \mu)^{1/2} / \{[(\mu' V^{-1} \mu)^{1/2} - [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{1/2}\}$

For $\alpha' > \alpha'^*$, the line would become (9) with α replaced by α', i.e., of form: $\sigma^*_{\alpha'} = (\mu' V^{-1} \mu)^{-1/2} \alpha'$- provided that the numerator of (7) is still non-negative and the same assets are included in the new optimal portfolio.

Yet, enduring a given (g $\sigma^*_{\alpha'}$), the maximum (α' g) is attained over (38), not over (9) - a different combination of assets would have to be optimal in the two problems, which would be a contradiction.

. Finally, we may replaced μ_0 by 0 and leave W_0 in the budget constraint with the role of a slack variable in the problem of section 3 and recover the unconstrained solution of section 2 – applying if the unused funds are used for consumption and there is no risk-less asset. The market line is still a straight line obeying conditions (27) to (38) with μ_0 replaced by 0 (but now $\nu^*_\alpha = 0$) while $W_0^* = g – L' V^{-1} \mu\ (\mu' V^{-1} \mu)^{-1} \alpha > 0$, i.e., for:

(58) $$\alpha' < (\mu' V^{-1} \mu) / (L' V^{-1} \mu) = \alpha'^{\#}$$

as stated in (16). While the condition holds, not all the budget g is exhausted to attain the given minimum variance achievable for a given α'. Above it – for $\alpha' > \alpha'^{\#}$ -, we fall on the solutions (49) to (53) – unless we can borrow at a zero interest rate. Graphically – in Fig. 1 -, the ray that intercepts the restricted line (53), does it at $\alpha'^{\#}$; its slope is then that of (9): $(\mu' V^{-1} \mu)^{1/2}$. ($\alpha'^{\#\#\#} < \alpha'^{\#} < \alpha'^{\#\#}$ and all must be higher than μ_0.)

It is arguable that if there is no risk-less asset at all, then, the bound would become $\alpha'^{\#\#}$ with μ_0 replaced by minus 1. For

$$\alpha' < [\mu' V^{-1} (\mu + L)] / [L' V^{-1} (\mu + L)] = \alpha'^{\#\#\#\#}$$

the solution of section 2 would hold with μ replaced by (μ + L) [13] and for α defined as return plus investment (i.e., replaced by α + g) - or that of section 3 with μ_0 replaced by minus 1, which is a legitimate "return" (total loss, discarding…) to holding unused wealth. For $\alpha' < \alpha'^{\#\#\#\#}$ - and, for positively sloped tangency to the curved market line (53) at that point, $\alpha'^{\#\#\#\#} < \alpha'^{\#}$ but, still, $\alpha'^{\#\#\#\#} > \alpha'^{\#\#\#}$ - we would optimize by "destroying" wealth. As this is not a reasonable behavior – even if one would think that definitely a lower bound for our expectations -, we would rather consume the unused resources as assumed - which requires a more complex framework to model.

. If there are borrowing constraints and - W_0 is bounded by - W_0', potentially larger than 0, if the restriction is binding, i.e., whenever for W* from (28), L' W* > g - W_0', the current optimal solutions would be valid with g replaced by g - W_0' and α by (α - μ_0 W_0') – in formulations with α, not α'.

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[13] See footnote 11.

5. Full Borrowing and Pure Risk. An interesting special case is that of a complete borrower, for which g = 0 and that chooses, therefore, – $W_0 > 0$. Then, $L'W^* = -W_0^*$, and expressions with α hold – (27) to (32), (34), (37) – with g replaced by 0... Yet, the equation that drives the market line, (38), becomes a ray with the same slope as the one before:

(59) $$\sigma_\alpha^* = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1/2} \alpha$$

This case is interesting in macroeconomic terms because it internalizes more explicitly the opportunity cost of capital, μ_0 – which would be endogenous at the macro level: paid to more risk-averse lenders by less risk-averse gamblers/borrowers (with the equilibrium μ_0 insuring equality between total, aggregate demand and supply…).

Income α is obtainable without any funds, and solely by bearing risk… As

(60) $$\alpha = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{1/2} \sigma_\alpha^*$$

$[(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{1/2}$ is the expected *price of pure risk*, the expected yield of one standard deviation of return volatility – it coincides with the slope of the general market line in space ($\sigma_{\alpha'}$, α'), of the inverse function of (38). On the other hand, $\mu' W^* = \alpha - \mu_0 W_0^*$ becomes the observed portfolio return involving the (borrowed) "investment" $-W_0^*$; then the expected return of an unitary application, $\frac{\mu' W_\alpha^*}{L' W_\alpha^*} = \frac{\mu' W_\alpha^*}{-W_0^*}$, is still constant (using (34)):

(61) $$\frac{\mu' W_\alpha^*}{L' W_\alpha^*} = \frac{\mu' W_\alpha^*}{-W_0^*} = \frac{\mu' V^{-1}(\mu - \mu_0 L)}{L' V^{-1}(\mu - \mu_0 L)} = r^*$$

as $(r^* - \mu_0)(- W_0^*) = \alpha$, replacing in (59), we obtain a form analogous to (38),

(62) $$\sigma_\alpha^* = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1/2} (r^* - \mu_0)(- W_0^*)$$

but, in fact, endogenously determined and independent of α or r*. Also:

(63) $$\sigma_{r^*}^* = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1/2} \left[\frac{\mu' V^{-1}(\mu - \mu_0 L)}{L' V^{-1}(\mu - \mu_0 L)} - \mu_0\right] =$$

$$= \frac{[(\mu - \mu_0 L)' V^{-1}(\mu - \mu_0 L)]^{1/2}}{L' V^{-1}(\mu - \mu_0 L)}$$

We note that r* equals $\alpha'^{\#\#}$ of (36) and $\sigma_{r^*}^* = \sigma_{\alpha'^{\#\#}}^*$ of (39).

Now, it is as if g became costly and endogenous… With no cost, we return to the first problem…

. If W_0 is bounded by $- W_0'$, if the restriction becomes active, i.e., whenever for W* from (28) evaluated at g = 0, $L' W^* > - W_0'$:

(64) $L' W_\alpha^* = L' V^{-1} (\mu - \mu_0 L) [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} \alpha > - W_0'$

that is, for:

(65) $\alpha > - W_0' [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)] / [L' V^{-1} (\mu - \mu_0 L)]$

or

(66) $\alpha /(- W_0') > [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)] / [L' V^{-1} (\mu - \mu_0 L)]$

we would subscribe to the structure of the problem of section 4 and the optimal solutions therein would be valid for g replaced by - W_0' and α by (α - μ_0 W_0') – in formulations with α, not α'.

. Finally, consider an institution manages assets for which positive W_i's contained in vector W_L are possible, and that also has (potential) creditors ("wealth" depositors) for which negative W_i's in W_B are possible such that it is choosing in the problem of section 3 (or of section 2, if $\mu_0 = 0$) a vector W = $(W_L, W_B)'$ compatible with those restrictions. Provided that Cov(u_L, u_B) is (a matrix of) 0(s) - V is block diagonal: V = diag(V_L, V_B) -, it is still true that the asset composition $W_L^*/(L'W_L^*) = \dfrac{V_L^{-1}(\mu_L - \mu_0 L)}{L'V_L^{-1}(\mu_L - \mu_0 L)}$ is going to be independent of α and that the expected return on the optimal asset applications would be $\dfrac{\mu_L ' V_L^{-1}(\mu_L - \mu_0 L)}{L'V_L^{-1}(\mu_L - \mu_0 L)}$. For the solution to be possible for g = 0, (34) > 0 (more deposits than loans) requires that $L' V_L^{-1} (\mu_L - \mu_0 L) < - L' V_B^{-1} (\mu_B - \mu_0 L)$.

6. Market Betas. Consider a general portfolio G, W^G, containing a fraction W_i^G of (risky) security i and no risk-less asset, and the assumptions of section 3. Then, its expected return is $\mu' W^G = W^{G'} \mu = R^G$. The covariance between its return and that of the optimal portfolio (for the range where it combines with the risk-free asset) is

(67) $$\mathrm{Cov}(u'W^G, u' W_\alpha^*) = W^{G'} V W_\alpha^* =$$

$$= W^{G'} V V^{-1} (\mu - \mu_0 L) [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g) =$$

$$= W^{G'} (\mu - \mu_0 L) [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g) =$$

$$= (R^G - \mu_0 W^{G'}L) [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g) =$$

$$= (R^G - \mu_0 W^{G'}L) \frac{\lambda_\alpha^*}{2} = (R^G - \mu_0 W^{G'}L) \frac{\sigma_\alpha^{*2}}{\alpha - g\mu_0}$$

or

$$(68) \qquad R^G - \mu_0 W^{G\prime}L = (\alpha - \mu_0 g) \frac{Cov(u'W^G, u'W^*_\alpha)}{\sigma^{*2}_\alpha}$$

An application of the asset $W^{G\prime}L = g$, equivalent to that for which the optimal portfolio was derived, could be replaced in the expression. Or confronting unitary applications, i.e., imposing g = 1, and $W^{G\prime}L = 1$. Notice however, that $W^{G\prime}L = 1$ restricts the current portfolio, unlike the efficient one, to include no risk-less asset – but R^G should be added by the same amount to reproduce the return to G. (68) becomes:

$$(69) \qquad R^G - \mu_0 = (\alpha' - \mu_0) \frac{Cov(u'W^G, u'W^*_{\alpha'})}{\sigma^{*2}_{\alpha'}}$$

Then, for any isolated (risky) asset i in amount u_i with expected value $W^{i\prime} \mu = R^i$, where W^i denotes a column-vector of 0's except on row i – including the case where u_i denotes the total amount of asset i in the market, $E[u_i] = R^i = \mu_i$.

$$(70) \qquad Cov(u_i, u'W^*_\alpha) = (R^i - \mu_0) \; [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)]^{-1} (\alpha - \mu_0 g) =$$

$$= (R^i - \mu_0) \frac{\lambda^*_\alpha}{2} = (R^i - \mu_0) \frac{\sigma^{*2}_\alpha}{\alpha - g\mu_0}$$

or $$R^i - \mu_0 = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)] \frac{Cov(u_i, u'W^*_\alpha)}{\alpha - \mu_0 g}$$

$$= 2 \frac{Cov(u_i, u'W^*_\alpha)}{\lambda^*_\alpha} = (\alpha - \mu_0 g) \frac{Cov(u_i, u'W^*_\alpha)}{\sigma^{*2}_\alpha}$$

For g =1 – with reference to the unit application on of the efficient portfolio -, the last expression reproduces the configuration of the "security market line" [14]:

$$(71) \qquad R^i - \mu_0 = (\alpha - \mu_0) \frac{Cov(u_i, u'W^*_{\alpha'})}{\sigma^{*2}_{\alpha'}}$$

Define the *market portfolio*, M, as the one including all of its securities and recover the μ_i's as denoting the aggregate return for asset i. Then:

$$(72) \qquad (\alpha - \mu_0 g) = (R^M - \mu_0 W^{M\prime}L) \frac{\sigma^{*2}_\alpha}{Cov(u'L, u'W^*_\alpha)}$$

or $$(\alpha - \mu_0 g) = [(\mu - \mu_0 L)' V^{-1} (\mu - \mu_0 L)] \frac{Cov(u'L, u'W^*_\alpha)}{R_M - \mu_0 L'W^M}$$

Then, for one unit of an isolated asset:

$$(73) \qquad (R^i - \mu_0) = (R^M - \mu_0 W^{M\prime}L) \frac{Cov(u_i, u'W^*_\alpha)}{Cov(u'L, u'W^*_\alpha)}$$

14 See Merton (1982), and Sharpe (1964) to whom it is there attributed.

W^{M}'L does not include the allocation of the risk-less asset, but neither does R^{M}. With its addition:

(74) $$(R^{i} - \mu_0) = (R^{M*} - \mu_0)\ \frac{Cov(u_i, u^{*}'W_{\alpha}^{*})}{Cov(u^{*}'L, u^{*}'W_{\alpha}^{*})}$$

where R^{M*} refers the expected return per monetary unit. The expected return of an existing asset in the market portfolio in excess of the return of the risk-free asset is a fraction of the market overall per unit excess return over the risk-free asset's times the ratio of the covariance of i's return with any optimal portfolio divided by the covariance of the market portfolio's return with that of the same optimal portfolio [15].

. Let us now consider the same type of manipulation but for portfolios generated under constrained borrowing – i.e., under conditions of section 4. Then, using (43), the covariance between the returns of portfolio G and an optimal portfolio is:

(75) $$Cov(u'W^{G}, u'W_{\alpha}^{*}) = W^{G}{'}VW_{\alpha}^{*} = \frac{1}{2}W^{G}{'}(\lambda\,\mu + \nu\,L) = \frac{1}{2}(R^{G}\,\lambda^{*} + \nu^{*}\,W^{G}{'}L)$$

Let G involve a unitary application, such that $W^{G}{'}L = 1$, and the market portfolio denote the unitary application as well. It is then easy to infer using (49) and (50) that:

(76) $$R^{G} = (\alpha' - \frac{2\sigma_{\alpha'}^{*\,2}}{\lambda_{\alpha'}^{*}}) + 2\,\frac{Cov(u'W^{G}, u'W_{\alpha'}^{*})}{\lambda_{\alpha'}^{*}}$$

$\lambda_{\alpha'}^{*}$ represents $\frac{\partial\sigma_{\alpha'}^{*\,2}}{\partial\alpha'} = 2\ \sigma_{\alpha'}^{*}\ \frac{\partial\sigma_{\alpha'}^{*}}{\partial\alpha'}$, where $\frac{\partial\sigma_{\alpha'}^{*}}{\partial\alpha'}$ refers the slope of the (convex) market line at α'. That is:

(77) $$R^{G} = (\alpha' - \frac{\sigma_{\alpha'}^{*}}{\frac{\partial\sigma_{\alpha'}^{*}}{\partial\alpha'}}) + \frac{Cov(u'W^{G}, u'W_{\alpha'}^{*})}{\sigma_{\alpha'}^{*}\,\frac{\partial\sigma_{\alpha'}^{*}}{\partial\alpha'}}$$

Also:

(78) $$R^{G} > \alpha' \quad \text{iff} \quad Cov(u'W^{G}, u'W_{\alpha'}^{*}) > \sigma_{\alpha'}^{*\,2}$$

[15] We did not inspect pricing conditions under which, in equilibrium, all market securities will be held – that would require introducing consumers, price of securities, decreasing its expected value, and, if there are no left-out securities of the optimal portfolio, eventually lead to W* = L – that is, the market and efficient portfolios coincide. That was beyond the scope of this short note.

Bibliography and References.

Black, F. (1972) "Capital Market Equilibrium with Restricted Borrowing." *Journal of Business.* Vol 45: 444-455.

Brealey, Richard A. and Stewart C. Myers. (2003) *Principles of Corporate Finance.* McGraw-Hill Higher Education. 7th Edition.

Campbell, John Y. Andrew W. Lo, and A. Craig MacKinlay. (1997) *The Econometrics of Financial Markets.* Princeton.

Cuthbertson, Keith. (1996) *Quantitative Financial Economics.* Wiley.

Dhrymes, Phoebus J. (1978) *Introductory Econometrics.* Springer-Verlag.

Duffie, Darrell. (1991) "The Theory of Value in Security Markets." In *Handbook of Mathematical Economics.* Vol. IV. Edited by Werner Hildenbrand and Hugo Sonnenschein. Amsterdam: North-Holland.

Hirshleifer, Jack and John G. Riley. (1992) *The Analytics of Uncertainty and Information.* Cambridge University Press.

Intriligator, Michael D. (1971) *Mathematical Optimization and Economic Theory.* Prentice-Hall.

Kancs, d'Artis and Kielyte, Julda. (2010) "European Integration and Labour Migration," *European Integration online Papers*, vol. 14, November.

Kancs, d'Artis. (2010) "Labour Migration in the Enlarged EU: A New Economic Geography Approach," *Journal of Economic Policy Reform*, 2011 (14): 171-188.

Kielyte, Julda. (2008) "Estimating panel data models in the presence of endogeneity and selection." *Journal of Economics and Econometrics* 51, 1–19.

Laffont, Jean-Jacques. (1989) *The Economics of Uncertainty and Information.* Massachusetts Institute of Technology.

Lange, Kenneth. (2004) *Optimization.* Springer.

Lintner, J. (1965) "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics.* Vol 47: 13-37.

Markowitz, H. (1952) "Portfolio Selection." *Journal of Finance.* Vol 7: 77-91.

Markowitz, H. (1959) *Portfolio Selection.* Cowles Foundation Monograph N. 16. New York: John Wiley & Sons, Inc.

Martins, Ana Paula. (1996) "Migration and Mobility: An Analytical Survey with Some Extensions." EERI Research Paper Series, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (1998) "A Nash-based Union's Utility Function with an Application to Portuguese Aggregate Data." EERI Research Paper Series, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (2002) "Reconciling the Nash and Kalai-Smorodinsky Cooperative Solutions: Generalized Maximands of CES Form." EERI Research Paper Series No 2002/05, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (2006) "On Simple Conditions for Mixed Equilibria in Dualistic Models. Part I: Degree of Mobility." EERI Research Paper Series No 2006/03, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (2006) "On Simple Conditions for Mixed Equilibria in Dualistic Models. Part II: Degree of Coverage." EERI Research Paper Series No 2006/04, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (2007) "On Depth and Retrospect: “I Forget, and Forgive – but I Discount." EERI Research Paper Series No 2007/07, Economics and Econometrics Research Institute, Brussels.

Martins, Ana Paula. (2010) "Frontier Techniques: Contrasting the Performance of (Single-) Truncated Order Regression Methods and Replicated Moments." *Journal of Economics and Econometrics*, Economics and Econometrics Research Institute (EERI), vol. 53(2), pages 75-93.

Martins, Ana Paula. (2010) "Splitting Games: Nash Equilibrium and the Optimisation Problem." *Journal of Economics and Econometrics*, Economics and Econometrics Research Institute (EERI), vol. 53(2), pages 1-28.

Martins, Ana Paula. (2011) "Compliance with the Institutional Wage in Dualistic Models." *Journal of Economics and Econometrics*, Economics and Econometrics Research Institute (EERI), vol. 54(2), pages 93-126.

Martins, Ana Paula. (2012) "Unemployment Insurance and Union Behavior: Comparison of Some Paradigms and Endogenous Membership." *Journal of Economics and Econometrics*, Economics and Econometrics Research Institute (EERI), vol. 55(1), pages 19-38.

Martins, Ana Paula. (2012) "Segmented Life-cycle Labor Markets – Portuguese Evidence." *Journal of Economics and Econometrics*, Economics and Econometrics Research Institute (EERI), vol. 55(1), pages 33-74.

Martins, Ana Paula. (2012) "Portfolio Selection – A Technical Note." EERI Research Paper Series No 2012/17, Economics and Econometrics Research Institute, Brussels.

Merton, Robert C. (1982) "On the Microeconomic Theory of Investment under Uncertainty." In *Handbook of Mathematical Economics.* Vol. II. Edited by Kenneth J. Arrow and Michael D. Intriligator. Amsterdam: North-Holland.

Sharpe, William. (1964) "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance.* Vol 19: 425-442.

Taha, Hamdy A. (1982) *Operations Research: An Introduction.* 3rd Edition. Macmillan.

Tobin, J. (1958) "Liquidity Preference as Behavior Towards Risk." *Review of Economic Studies.* Vol 25: 68-85.

Wilmott, Paul. (2001) *Paul Wilmott Introduces Quantitative Finance.* Wiley.

Appendix

Assume that the function to be minimized has also a linear part in W – that is, let C be a (1xn) vector; then C W is added to W'VW; we require V to be symmetric and – for SOC for a minimum to be satisfied – positive definite. We admit that m (independent) linear restrictions must be satisfied, m < n: α is an (mx1) column-vector, μ an (n x m) matrix. We want to:

$$\min_{W} \quad W' V W + C W$$

$$\text{s.t.} \quad \mu' W \geq \alpha$$
$$W_i \geq 0$$

Denoting by λ an m-row (mx1) column vector containing the Lagrange multipliers, we can write:

$$\min_{W,\lambda} \; L(W, \lambda) = W' V W + C W + \lambda' (\alpha - \mu' W)$$

where λ denotes the multiplier. F.O.C. imply:

$$\frac{\partial L}{\partial W} = 2 W' V + C' - \lambda' \mu' = 0 \text{ (a (1 x n) vector)}$$

$$\frac{\partial L}{\partial \lambda'} = \alpha - \mu' W = 0 \text{ (an (m x 1) vector)}$$

Then, one can show, using similar steps as in the text, that:

$\lambda^* = (\mu' V^{-1} \mu)^{-1} (2 \alpha + \mu' V^{-1} C')$

$W^* = (V^{-1}/2) [\mu (\mu' V^{-1} \mu)^{-1} (2 \alpha + \mu' V^{-1} C') - C']$

$W^{*\prime}VW^* + CW^* = (1/4) [(2 \alpha + \mu'V^{-1}C')' (\mu'V^{-1}\mu)^{-1}(2 \alpha + \mu'V^{-1}C') - CV^{-1}C']$