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Frontier Techniques: Contrasting the Performance of (Single-)Truncated Order Regression Methods and Replicated Moments

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ABSTRACT

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This research contrasts three econometric alternatives for stochastic efficiency frontier analysis: order – inter-quantile – and inverse order regression under the assumption of truncated error term distribution, and replicated moment estimation.

The demonstration departs from a simple linear regression form of the effective frontier; truncated (at zero) errors are then added to it for simulation purposes. For order regression, experiments with the standard normal, uniform, exponential, Cauchy and logistic error terms are provided. For complex error structures we rely on normal distributions only.

The three alternatives would perform satisfactorily for simple error disturbances, especially if they are normal. With more than one residual added to the dependent variable, the weight of the unrestricted range one can blur the conclusions regarding observation efficiency.

JEL Classification: C24; C10.

Keywords: Stochastic Frontier Model. Generalized Method of Order Statistics; Minimum Distance Method of Order Statistics. Inverse Order Regression. Replicated Moments. Linear Models.

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Introduction

Efficiency analysis is *ex-post* tested with the use of a variety of quantitative methods [1]. These sometimes rely on the estimation of a efficient frontier, with distance to it being an indicator of the observation performance. It is the purpose of this note to compare the results of three efficiency frontier estimators: straightforward least squares, adding the minimum or subtracting the maximum estimated error – according to whether the efficiency being measured is, say, cost or revenue - to the estimated model residuals, the method of order statistics towards a truncated error distribution and a replicated moment one.

The analysis relies on simulation, departing from a simple regression model to both illustrate and compare the performance of the methods. Two environments are staged: in one, a truncated at zero normal, uniform, exponential, Cauchy and logistic error terms are then to a deterministic linear model, providing a simple linear regression departure. In a second attempt, an extra normal untruncated random error is additionally included.

For all series, methods evaluate both a lower as an upper truncation hypothesis and none at all. We would hope that the true assumption would emerge with the best performance.

In the order – interquantile – estimation we considered only three alternatives for the null hypothesis: the normal, the exponential, and the uniform itself. The method rely on a two step estimation procedure, departing from rankings of the (first step) OLS residual estimates [2]. It suggests direct inference, and an indirect approach relying on inverting the direct form. The latter is also subject to estimation by the method of moments in a "replicated" version.

Replicated moment estimation was previously forwarded in the literature [3]. One can justify it in linear regression if we note that for a model with k parameters we have in fact nxk statistics – k, the dependent variable and k-1 independent variables, for each observation. We essayed with the straight-forward replication, with a weighted least squares and a generalized least squares one.

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[1] See Koop (2003), p. 147 and 168-177, for references on stochastic frontier modelling; Murillo-Zamorano (2004), for a recent survey of both non-parametric – as Data Envelope Analysis – as parametric and stochastic – as in the present research; and Greene (1997).

[2] See Martins (2005).

[3] Martins (2003).

For simplicity, efficiency is modelled additively and the simulations depart from a linear deterministic counterpart. The methods can easily be adapted to apply to nonlinear frontiers, the logarithm of which could be subject to the proposed procedures; inefficiency would then be multiplicative towards the deterministic optimum.

The exposition proceeds as follows: section 1 describes the generated basic random series used in the simulations, summarizing briefly theoretical foundations behind the data generating procedures under truncated assumptions. Section 2 applies the "method of order statistics" [4] – in minimum distance versions -, an inter-quantilic inference method. The inverse approach is forwarded in section 3. Section 4 illustrates results from replicated method of moments applied to the pillar equation the last procedure. Some concluding remarks end the research.

1. Data and Data Generating Procedures

. The generation of random samples started by independent draws from the uniform distribution [5], W_i's, inverted according to the required cdf – a procedure justified by a well-known:

Theorem. For any random variable Z with uniform distribution in the (0, 1) interval – i.e., with cdf U(z) = z and pdf u(z) = U'(z) = 1, for 0 < z < 1 -, $X = F^{-1}(Z)$ – where F(z) exhibits properties of a cdf in the appropriate domain of z and $F^{-1}(x)$ denotes the inverse function of F(x) - has pdf f(x) = F'(x), and cdf F(X).

Proof: Using variable transformation, the pdf of $X = G(Z) = F^{-1}(Z)$, which implies, Z = F(X) $= G^{-1}(X)$, is $u[G^{-1}(x)]\ \frac{dG^{-1}(x)}{dx}$, with $z = G^{-1}(x) = F(x)$; then, it equals 1 $\frac{dF(x)}{dx}$ = F'(x) = f(x).

Corollary: If X is distributed according to a cdf F(X), $W = H^{-1}[F(X)]$ has cumulative distribution function H(w), provided that H(w) is an appropriate cdf.

Then, testing the validity of a distribution function F(.), for a random iid sample, is equivalent to test if $W_k = H^{-1}(Z_k) = H^{-1}[F(X_k)]$ have distribution function H(.).

For example, testing that X_k come from the cdf F(x) is equivalent to test – using the same sample…- if the transformed sample values $W_k = \Phi^{-1}[F(X_k)]$, where $\Phi(.)$ denotes the cdf of the standard normal, come from the standard normal.

Three uniform (0, 1) random [6] series (of size 100 each) were created.

[4] Alternatively, we would name it rank regression… Yet, rank estimation seems a term more closely (and already) associated to robust – non-parametric – methods in the econometrics literature.

[5] Using the RAND(.) function of EXCEL.

[6] We use the same series as in Martins (2005).

One was used to provide the exogenous variable X, with normal distribution of mean 7 and 3 standard deviations: $X_i = 7 + 3\ \Phi^{-1}(W_i)$, where $\Phi(z)$ denotes the cumulative standard normal and $\Phi^{-1}(z)$ its inverse.

One of the other series was (invariably) used to create the residuals, E, to form the dependent variable, Y, of our baseline (simple) regression model, always created as

$$(1.1) \qquad Y_i = 5 + 6 X_i + E_i$$

The E_i's are eventually generated by a truncated (at zero) distribution.

Finally, we experimented adding an unrestricted standard normal residual, V_i, built by using the inverse normal of the third uniform random series that would appear as:

$$(1.2) \qquad Y_i = 5 + 6 X_i + E_i + V_i$$

This is the formulation usually encountered in stochastic frontier modelling.

. Let a be the lower truncation point and b the upper truncation point of a given cdf F(e). We rely on the fact that being W_i, uniform, it should equal the truncated cdf according to the definition:

$$(1.3) \qquad F[e \mid a< e < b] = \frac{F(e)-F(a)}{F(b)-F(a)} \quad , \quad a < e < b$$

Obviously, the problem could interchangeably have been defined in terms of the limiting probabilities, i.e., note that a = $F^{-1}(\alpha)$ and b = $F^{-1}(\beta)$ - i.e., for α - the probability left-out to the left of the original F(e) - replaced by α = F(a) and β - with 1 - β being the upper portion of the original distribution that was cut-off - by β = F(b).

$$(1.4) \qquad F[e \mid F^{-1}(\alpha) < e < F^{-1}(\beta)] = \frac{F(e)-\alpha}{\beta-\alpha} \quad , \quad F^{-1}(\alpha) < e < F^{-1}(\beta)$$

The pdf obeys

$$(1.5) \qquad f[e \mid a< e < b] = \frac{f(e)}{F(b)-F(a)} = \frac{f(e)}{\beta-\alpha} \quad , \quad F^{-1}(\alpha) = a < e < b = F^{-1}(\beta)$$

Say, an efficiency production frontier model would specify the addition to its deterministic – efficient - counterpart of an error e such that F[e | - ∞ < e < 0]; a minimum cost boundary model, F[e | 0 < e < ∞].

We start from our uniform (0,1) series, W_i. We postulate that:

(1.6) $$W_i = \frac{F(E_i)-F(a)}{F(b)-F(a)} = \frac{F(E_i)-\alpha}{\beta-\alpha}$$

Therefore, after choosing F(.) and the truncation limits, we can generate a series of a truncated distribution considering:

(1.7) $$E_i = F^{-1}\{W_i\,[F(b) - F(a)] + F(a)\} = F^{-1}[W_i\,(\beta - \alpha) + \alpha\,]$$

And of course $a < E_i < b$. We considered three cases, of general form $E_i = F^{-1}\{F(a)+[F(b)-F(a)]W_i\}$.

For a standard distribution at $b = \infty$, b is replaced by ∞ and a by $-\infty$: $E_i = F^{-1}(W_i)$.

For a truncated above distribution at $b = 0$, b is replaced by 0 and a by $-\infty$: $E_i = F^{-1}[W_i\,F(0)]$.

For a truncated below distribution at $b = \infty$, a is replaced by 0 and a by 0: $E_i = F^{-1}\{W_i\,[1 - F(0)] + F(0)\}$.

. We depart from several hypothesis concerning the error term distribution to be added to the deterministic part of the model: standard normal, uniform, exponential, Cauchy and logistic; the error terms other than the normal were further transformed so as to generate for the standard series a null expected value and unitary variance one; for the Cauchy the series were multiplied 0.67449037, the inverse standard deviation of the zero-mean normal with the same quartiles. Samples had always size n = 100. In the philosophy of the estimated regressions, these distributions are then a function of the regression residual divided by a standard error, also subject to estimation [7].

We report below, in I. – V., for each case the generic distribution used, the procedures taken to generate the truncated series, and tables with a summary of the descriptive statistics of the created input error series, E_i, of the output series, Y_i, and the OLS results of the regression of Y_i on X_i. $\hat{a}_1$ and $\hat{a}_2$ denote, respectively, the intercept and slope estimates of the linear model, $\hat{E}_i$, the estimated OLS residuals; SD refers standard deviation; as usual, a bar indicates a mean, except for the adjusted R^2, $\overline{R}^2$. The first table also contains information on X_i, and the "mother" uniform (0, 1) random series, W_i – the cumulative distribution function - from which all error series were created.

Two types of series were created. Tables 1.1.1 to 1.1.5 correspond to the simple form (1.1); Tables 1.2.1 to 1.2.5 to form (1.2) – the previous Y_i series are added of an extra standard normal error, V_i.

[7] In our case, 1 is always the expected value of the estimates of this parameter. But we could have multiply our theoretical error terms by some other factor before addition to the deterministic model.

I. Standard Normal, F(e) = $\Phi(e)$, -∞ < e < ∞. F(0) = $\frac{1}{2}$.

The E_i's come from the normal – $\Phi(z)$ denotes the standard normal. Then:

1. $E_i = \mu + \sigma\ \Phi^{-1}(W_i)$, $-\infty < E_i < \infty$ (i.e., F(a) = α = 0, F(b) = β = 1).

2. a truncated above normal at $\frac{1}{2}$ (F(a) = α = 0, F(a) = β = $\frac{1}{2}$). Then $E_i = \mu + \sigma\ \Phi^{-1}(\frac{1}{2}W_i)$ and $-\infty < E_i < \mu + \sigma\ \Phi^{-1}(\frac{1}{2})$.

3. a truncated from below normal at 50% (F(a) = α = $\frac{1}{2}$, F(a) = β = 1)[8]. Then $E_i = \mu + \sigma\ \Phi^{-1}(\frac{1}{2}+\frac{1}{2}W_i)$ and $\mu + \sigma\ \Phi^{-1}(\frac{1}{2}) < E_i < \infty$.

μ and σ were always fixed to 0 and 1 respectively (we rely on the standard normal). Other values for σ could have been used instead: for our purposes, it is irrelevant - it is a parameter that will be subject to estimation inquiry.

[8] Notice that even if μ = 0, we generate a different sample than the symmetric of the first median one.

Table 1.1.1 Descriptive Statistics and Normal Distribution			
	$\Phi(z)$	$\Phi[z \mid z<0]$	$\Phi[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{X}$	6.93420253	6.93420253	6.93420253
SD_X	3.42581545	3.42581545	3.42581545
$\overline{W}$	0.48738268	0.48738268	0.48738268
SD_W	0.29383280	0.29383280	0.29383280
Min W	0.004137	0.004137	0.004137
Max W	0.971564	0.971564	0.971564
$\overline{E}$	-0.04207797	-0.81551	0.766033
SD_E	0.96388894	0.590691	0.5728
Min E	-2.64066	-2.86753	0.005185
Max E	1.904291	-0.03565	2.191209
$\overline{Y}$	46.56313726	45.7897	47.37125
SD_Y	20.59755954	20.5325	20.61684
Min Y	-9.61574	-10.2641	-8.91377
Max Y	105.5175	104.0748	105.8742
OLS:			
$\hat{a}_1$	4.91721513	4.246969	5.656605
$SD_{\hat{a}_1}$	0.21955834	0.134395	0.12992
$\hat{a}_2$	6.00587045	5.990989	6.015781
$SD_{\hat{a}_2}$	0.02841551	0.017394	0.016814
$\hat{\sigma}$	0.96858336	0.592885	0.573145
Min $\hat{E}$	-2.61678	-2.02408	-0.80977
Max $\hat{E}$	1.943501	0.836249	1.419156
e’e	91.9390	34.4483	32.1925
R^2	0.99781106	0.9991746	0.999235
$\overline{R}^2$	0.99778873	0.9991662	0.999227

From our exercise, for example, we conclude that for standard normals, the results reported in Table 1.1.1 for $\overline{E}$ and SD_E are the approximations of the mean and standard deviation of the truncated distributions [9]. The OLS estimate of the intercept, a_1, varies across the three columns – i.e., samples -, capturing the bias induced by truncation – the truncated errors do not have a zero mean any longer. Also, a reduced estimated standard error is captured in the two truncated samples. This pattern is reproduced in all the 5 cases.

Table 1.2.1 Descriptive Statistics and Normal Distribution			
	$\Phi(z)$	$\Phi[z \mid z<0]$	$\Phi[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{E}$	-0.08606	-0.85949	0.722054
SD_E	1.391352	1.19157	1.153271
Min E	-3.73467	-3.96154	-1.72338
Max E	3.243231	1.807427	3.601153
$\overline{Y}$	46.51916	45.74572	47.32727
SD_Y	20.73665	20.67358	20.75569
Min Y	-11.2415	-11.8899	-10.5395
Max Y	105.2068	103.5069	105.5229
OLS:			
$\hat{a}_1$	4.639729	3.969483	5.379118
$SD_{\hat{a}_1}$	0.31549	0.270796	0.259903
$\hat{a}_2$	6.039545	6.024664	6.049456
$SD_{\hat{a}_2}$	0.040831	0.035047	0.033637
$\hat{\sigma}$	1.391788	1.194619	1.146563
Min $\hat{E}$	-3.77121	-3.1785	-2.61471
Max $\hat{E}$	3.448995	2.74158	3.028807
e'e	189.8331	139.8573	128.8315
R^2	0.995541	0.996695	0.996979
$\overline{R}^2$	0.995495	0.996661	0.996948

With the doubled error structure, we confirm – see column 1 of Table 1.2.1 – an increase in the standard error: the variance of the error doubled and the standard error becomes 1.4…

[9] See, for example, Johnston, Kotz and Balakrishnan (1994), p. 159, Table 13.10 for comparable tabulations.

II. Uniform of mean 0 and 1 standard deviation. $F(e) = \frac{z}{2\sqrt{3}} + \frac{1}{2}$, $-\sqrt{3} < e < \sqrt{3}$. $F(0) = \frac{1}{2}$

.

1. $E_i = (W_i - \frac{1}{2})\ 2\sqrt{3}$, $-\sqrt{3} < E_i < \sqrt{3}$ (i.e., $F(a) = \alpha = 0$, $F(b) = \beta = 1$).

2. a truncated above error: $E_i = (W_i \frac{1}{2} - \frac{1}{2})\ 2\sqrt{3}$, $-\sqrt{3} < E_i < 0$. (($F(a) = \alpha = 0$, $F(a) = \beta = 0.5$.)

3. a truncated from below uniform at 50% ($F(a) = \alpha = 0.50$, $F(a) = \beta = 1$) [10]. Then $E_i = (W_i \frac{1}{2} + \frac{1}{2} - \frac{1}{2})\ 2\sqrt{3}$ and $0 < E_i < \sqrt{3}$.

[10] Notice that even if $\mu = 0$, we generate a different sample than the symmetric of the first quartile one.

Table 1.1.2. Uniform Distribution			
	$U(z)$	$U[z \mid z<0]$	$U[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	$-\sqrt{3}$	$-\sqrt{3}$	0
b	$\sqrt{3}$	0	$\sqrt{3}$
$\overline{E}$	-0.04371	-0.88788	0.844172
SD_E	1.017867	0.508933	0.508933
Min E	-1.71772	-1.72489	0.007166
Max E	1.633545	-0.04925	1.682798
$\overline{Y}$	46.56151	45.71734	47.44939
SD_Y	20.59925	20.57079	20.57079
Min Y	-9.64313	-10.4595	-8.72744
Max Y	105.6031	104.0309	105.763
OLS:			
$\hat{a}_1$	4.917432	4.092691	5.824741
$SD_{\hat{a}_1}$	0.231863	0.115931	0.115931
$\hat{a}_2$	6.005604	6.002802	6.002802
$SD_{\hat{a}_2}$	0.030008	0.015004	0.015004
$\hat{\sigma}$	1.022865	0.511432	0.511432
Min $\hat{E}$	-1.69138	-0.84569	-0.84569
Max $\hat{E}$	1.675857	0.837928	0.837928
e'e	102.533	25.6332	25.6332
R^2	0.997559	0.999388	0.999388
$\overline{R}^2$	0.997534	0.999382	0.999382

Table 1.2.2 Descriptive Statistics and Uniform Distribution			
	$U(z)$	$U[z \mid z<0]$	$U[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{E}$	-0.08769	-0.93186	0.800193
SD_E	1.424981	1.138822	1.138822
Min E	-3.25947	-3.41317	-1.68112
Max E	3.3325	1.76282	3.494871
$\overline{Y}$	46.51753	45.67336	47.40541
SD_Y	20.73803	20.71081	20.71081
Min Y	-11.2689	-12.0852	-10.3532
Max Y	105.1264	103.4826	105.2146
OLS:			
$\hat{a}_1$	4.639946	3.815204	5.547255
$SD_{\hat{a}_1}$	0.323208	0.257895	0.257895
$\hat{a}_2$	6.039279	6.036477	6.036477
$SD_{\hat{a}_2}$	0.04183	0.033377	0.033377
$\hat{\sigma}$	1.425833	1.137705	1.137705
Min $\hat{E}$	-3.30623	-2.60617	-2.60617
Max $\hat{E}$	3.539088	2.805097	2.805097
e'e	199.234	126.8486	126.8486
R^2	0.995321	0.997013	0.997013
$\overline{R}^2$	0.995273	0.996982	0.996982

III. Exponential of mean 0 and 1 standard deviation. F(e) = 1 – exp[-(e + 1)], - 1 < e < ∞. F(0) = 1 – exp(-1).

1. E_i = - ln(1 – W_i) -1, -1< E_i < ∞ (i.e., F(a) = α = 0, F(b) = β = 1).

2. a truncated above distribution E_i = - ln{1 – W_i [1 – exp(-1)]} - 1, -1 < E_i < 0. (F(a) = α = 0, F(a) = β = 1 - exp(-1).)

3. a truncated from below at 50% (F(a) = α = 1 - exp(-1), F(a) = β = 1) [11]. Then E_i = - ln{1 – W_i [exp(-1) + 1 – exp(-1)]} – 1 and 0 < E_i < ∞.

[11] Notice that even if μ = 0, we generate a different sample than the symmetric of the first quartile one.

Table 1.1.3. Exponential Distribution			
	$E(z)$	$E[z \mid z<0]$	$E[z \mid z>0]$
α	0	0	1 - exp(-1)
β	1	1 - exp(-1)	1
a	- 1	- 1	0
b	∞	0	∞
$\overline{E}$	-0.06005	-0.59309	0.93995
SD_E	0.872779	0.285386	0.872779
Min E	-0.99585	-0.99738	0.004146
Max E	2.560093	-0.04771	3.560093
$\overline{Y}$	46.54517	46.01213	47.54517
SD_Y	20.69529	20.57358	20.69529
Min Y	-9.90642	-10.1901	-8.90642
Max Y	105.5735	104.0436	106.5735
OLS:			
$\hat{a}_1$	4.692299	4.373077	5.692299
$SD_{\hat{a}_1}$	0.196885	0.064909	0.196885
$\hat{a}_2$	6.035714	6.00488	6.035714
$SD_{\hat{a}_2}$	0.025481	0.008401	0.025481
$\hat{\sigma}$	0.868558	0.286346	0.868558
Min $\hat{E}$	-1.12432	-0.41942	-1.12432
Max $\hat{E}$	2.604508	0.544894	2.604508
e’e	73.9305	8.03541	73.9305
R^2	0.998256	0.999808	0.998256
$\overline{R}^2$	0.998239	0.999806	0.998239

Table 1.2.3 Descriptive Statistics and Exponential Distribution			
	$E(z)$	$E[z \mid z < 0]$	$E[z \mid z > 0]$
α	0	0	1 - exp(-1)
β	1	1 - exp(-1)	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{E}$	-0.10403	-0.63707	0.895972
SD_E	1.301276	1.064439	1.301276
Min E	-2.7419	-2.96853	-1.7419
Max E	3.292368	1.77584	4.292368
$\overline{Y}$	46.50119	45.96815	47.50119
SD_Y	20.83193	20.71392	20.83193
Min Y	-11.5322	-11.8158	-10.5322
Max Y	105.5483	103.4869	106.5483
OLS:			
$\hat{a}_1$	4.414813	4.095591	5.414813
$SD_{\hat{a}_1}$	0.291485	0.240641	0.291485
$\hat{a}_2$	6.069389	6.038554	6.069389
$SD_{\hat{a}_2}$	0.037724	0.031144	0.037724
$\hat{\sigma}$	1.28589	1.061588	1.28589
Min $\hat{E}$	-2.87538	-2.31735	-2.87538
Max $\hat{E}$	3.606446	2.529614	3.606446
e'e	162.0444	110.443	162.0444
R^2	0.996228	0.9974	0.996228
$\overline{R}^2$	0.99619	0.997373	0.99619

IV. Cauchy. $F(e) = \frac{1}{2} + \frac{1}{\pi}\arctan(\frac{e}{0.67449037})$, $-\infty < e < \infty$. $F(0) = \frac{1}{2}$

1. $E_i = 0.67449037 \tan[\pi (W_i - \frac{1}{2})]$, $-\infty < E_i < \infty$ (i.e., $F(a) = \alpha = 0$, $F(b) = \beta = 1$).

2. truncated above ($F(a) = \alpha = 0$, $F(a) = \beta = 0.5$) $E_i = 0.67449037 \tan[\pi (W_i \frac{1}{2} - \frac{1}{2})]$ $-\infty < E_i < 0$.

3. a truncated from below at 50% (F(a) = α = 0.50, F(a) = β = 1) [12]. Then E_i = 0.67449037 $\tan\{\pi\ (W_i \frac{1}{2} + \frac{1}{2} - \frac{1}{2})\}$, and $0 < E_i < \infty$.

Table 1.1.4. Cauchy Distribution			
	$C(z)$	$C[z \mid z<0]$	$C[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	$-\infty$	$-\infty$	0
b	∞	0	∞
$\overline{E}$	-0.47443	-2.54858	1.59972
SD_E	5.551637	10.44655	2.61437
Min E	-51.8925	-103.789	0.004383
Max E	7.53004	-0.03015	15.09023
$\overline{Y}$	46.13078	44.05663	48.20494
SD_Y	20.9853	22.02287	21.21516
Min Y	-9.60473	-38.5841	-8.92749
Max Y	107.1165	104.0923	111.351
OLS:			
$\hat{a}_1$	5.162756	4.747173	5.57834
$SD_{\hat{a}_1}$	1.262814	2.366004	0.584441
$\hat{a}_2$	5.90811	5.668923	6.147296
$SD_{\hat{a}_2}$	0.163435	0.306211	0.075639
$\hat{\sigma}$	5.570912	10.43765	2.578264
Min $\hat{E}$	-51.1332	-100.214	-2.56961
Max $\hat{E}$	8.049362	5.627642	13.41855
e'e	3041.44	10676.6	651.449
R^2	0.930239	0.777644	0.98538
$\overline{R}^2$	0.929527	0.775376	0.985231

[12] Notice that even if μ = 0, we generate a different sample than the symmetric of the first quartile one.

Table 1.2.4 Descriptive Statistics and Cauchy Distribution			
	$C(z)$	$C[z\|z<0]$	$C[z\|z>0]$
α	0	0	0.50
β	1	0.50	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{E}$	-0.51841	-2.59256	1.555742
SD_E	5.710954	10.60107	2.677911
Min E	-52.9865	-104.883	-1.74019
Max E	7.131926	1.825145	14.69211
$\overline{Y}$	46.08681	44.01265	48.16096
SD_Y	21.14048	22.20365	21.33452
Min Y	-11.2305	-39.6781	-10.5532
Max Y	107.7512	103.5167	111.9857
OLS:			
$\hat{a}_1$	4.88527	4.469687	5.300853
$SD_{\hat{a}_1}$	1.300353	2.404099	0.593542
$\hat{a}_2$	5.941784	5.702598	6.180971
$SD_{\hat{a}_2}$	0.168293	0.311141	0.076817
$\hat{\sigma}$	5.736516	10.6057	2.618415
Min $\hat{E}$	-52.2876	-101.369	-3.91537
Max $\hat{E}$	7.678775	5.965316	13.04796
e'e	3224.947	11023.13	671.8975
R^2	0.927112	0.774149	0.985089
$\overline{R}^2$	0.926368	0.771845	0.984937

V. Logistic $F(e) = \dfrac{1}{1+e^{-\frac{\pi e}{\sqrt{3}}}}$, $-\infty < e < \infty$. $F(0) = \frac{1}{2}$.

1. $E_i = -\ln(\frac{1}{W_i} - 1)\frac{\sqrt{3}}{\pi}$, $-\infty < E_i < \infty$ (i.e., $F(a) = \alpha = 0$, $F(b) = \beta = 1$).

2. $E_i = -\ln(\frac{1}{W_i \frac{1}{2}} - 1)\frac{\sqrt{3}}{\pi}$, $-\infty < E_i < 0$. ($F(a) = \alpha = 0$, $F(a) = \beta = 0.5$)

3. $E_i = -\ln(\frac{1}{W_i \frac{1}{2} + \frac{1}{2}} - 1)\frac{\sqrt{3}}{\pi}$, and $0 < E_i < \infty$. ($F(a) = \alpha = 0.5$, $F(a) = \beta = 1$)

Table 1.1.5. Logistic Distribution			
	$L(z)$	$L[z \mid z < 0]$	$L[z \mid z > 0]$
α	0	0	0.50
β	1	0.50	1
a	$-\infty$	$-\infty$	0
b	∞	0	∞
$\overline{E}$	-0.04218	-0.77781	0.726205
SD_E	0.933197	0.617457	0.583065
Min E	-3.02327	-3.40657	0.004562
Max E	1.946877	-0.03136	2.337039
$\overline{Y}$	46.56304	45.8274	47.33142
SD_Y	20.59414	20.51993	20.62961
Min Y	-9.60711	-10.1925	-8.97949
Max Y	105.451	104.0887	105.8605
OLS:			
$\hat{a}_1$	4.921187	4.311658	5.591442
$SD_{\hat{a}_1}$	0.212574	0.140316	0.131973
$\hat{a}_2$	6.005283	5.987097	6.019435
$SD_{\hat{a}_2}$	0.027512	0.01816	0.01708
$\hat{\sigma}$	0.93777	0.619007	0.582199
Min $\hat{E}$	-2.99747	-2.58876	-0.8008
Max $\hat{E}$	1.986474	0.842755	1.602047
e’e	86.1824	37.5506	33.2176
R^2	0.997947	0.999099	0.999212
$\overline{R}^2$	0.997926	0.99909	0.999204

Table 1.2.5 Descriptive Statistics and Logistic Distribution			
	$L(z)$	$L[z \mid z<0]$	$L[z \mid z>0]$
α	0	0	0.50
β	1	0.50	1
a	- ∞	- ∞	0
b	∞	0	∞
$\overline{E}$	-0.08616	-0.82179	0.682227
SD_E	1.3736	1.211962	1.153544
Min E	-4.11728	-4.50058	-1.73672
Max E	3.176014	1.821518	3.585969
$\overline{Y}$	46.51906	45.78342	47.28744
SD_Y	20.73348	20.66151	20.7681
Min Y	-11.2329	-11.8182	-10.6052
Max Y	105.1836	103.5146	105.5803
OLS:			
$\hat{a}_1$	4.6437	4.034172	5.313956
$SD_{\hat{a}_1}$	0.311471	0.275649	0.259526
$\hat{a}_2$	6.038958	6.020772	6.053109
$SD_{\hat{a}_2}$	0.040311	0.035675	0.033588
$\hat{\sigma}$	1.374058	1.216028	1.144903
Min $\hat{E}$	-4.1519	-3.74318	-2.60073
Max $\hat{E}$	3.380102	2.706188	3.06451
e'e	185.0274	144.915	128.4586
R^2	0.995652	0.996571	0.996992
$\overline{R}^2$	0.995608	0.996536	0.996961

. Our main objective of the following sections will be to obtain estimates of the linear model:

(1.8) $$Y_i = a_1 + a_2 X_i + E_i , \quad i = 1,2,..., 100$$

or

(1.9) $$Y_i = a_1 + a_2 X_i + E_i + V_i , \quad i = 1,2,..., 100$$

under the assumption that the E_i's have a truncated (-∞, 0) or (0, ∞) distribution or unrestricted for each of the fifteen series Y_i that were created.

Notice also that (1.8) and (1.9) are equivalent to:

(1.10) $Y_i = a_1' + a_2 X_i + RESI_i$, $i = 1,2,..., 100$

where $E[RESI_i] = 0$, $Var(RESI_i) = Var(E_i)$, or $VAR(E_i) + Var(V_i)$, with a_1' capturing bias of the estimation.

For each of the (2 times) fifteen Y series, OLS residuals were thus generated, $\hat{E}_i$. The ascending rank of each residual series was recorded as a variable R_i – the ranking of the order statistics of the estimated residual series. In general, and relying on well-known results, whatever the distribution, F(.) of the E_i's, we expect that $F(O_j) = \frac{j}{n+1} = \frac{j}{101}$ where O_j denotes the j-th order statistic. We create in accordance $S_i = \frac{R_i}{n+1} = \frac{R_i}{101}$. Then, we hope to approach $F(E_i)$ in inter-quantile or truncated estimation

We used TSP 4.4 [13] and EXCEL for computation. From the former, we relied more heavily on OLSQ LSQ (FRML, EQSUB) and matrix routines.

2. Order Estimation

. In section 1, we established the required principles to generate estimation strategies after a first-step OLS run:

For a truncated distribution at known truncation points, inter-quantile inference would rely on the adjustment by NLS for example of:

(2.1) $$S_i \approx \frac{F(\frac{Y_i - a_1 - a_2 X_i}{\sigma}) - F(a)}{F(b) - F(a)}$$

where $S_i = \frac{R_i}{n+1} = \frac{R_i}{101}$ denotes the rank of estimated residual i over the sample size plus 1. a and b are fixed to reproduce the three cases: $(-\infty, \infty)$, $(-\infty, 0)$ or $(0, \infty)$. Form (1.8) rather than (1.9) would appear to suggest (2.1) – with an error added to the right hand-side at the rankings approximation (2.1).

The null hypothesis distributions F(.) considered were the standard normal, the uniform ($-\sqrt{3}$, $\sqrt{3}$), and the transformed exponential $(-1, \infty)$. Given the way the series were built, we would hope to recover an estimate of 1 for σ in all cases of simple error structure for thr true distribution

[13] Hall and Cummins (1997) and (1998).

and 5 and 6 for the linear parameters, intercept and slope, if we stage the appropriately – to the sample - truncated cdf…

With respect to (2.1), "generalized" nonlinear least squares minimizing $e'W^{-1}e$, where e denote the difference of S_i minus the inferred $\frac{F(\frac{Y_i - a_1 - a_2 X_i}{\sigma}) - F(a)}{F(b) - F(a)}$, for a variance-covariance matrix W – inferring the variance-covariance matrix of the vector representing the right hand-side of (2.1) - given by

$$\text{(2.2)} \qquad W = \left[\frac{Min(S_i, S_j)[1 - Max(S_i, S_j)]}{n+2} \right]$$

off it would also be a possibility – that would simply extend the GMM – generalized method of moments - estimation principle [14] to a GMOS – general method of order statistics [15].

We present below only the minimum distance method of order-statistics (MDMOS) estimators of equation (2.1). Let F(θ, X) denote the vector of cdf functions for each observation j, F(θ, X_j), and S of the ranks over n + 1 deducted from the first step OLS regression. Then, MDMOS estimators are obtained from

$$\text{(2.3)} \qquad \underset{\hat{\theta}_{MDMOS}}{Min} \; [S - F(\theta, X)]' \, [S - F(\theta, X)]$$

which just requires applying (nonlinear) least squares to S = F(θ, X) + u, where u denotes a vector of residuals.

Standard errors were obtained according to MM principles [16]: letting G(θ) = $\frac{\partial F(\theta, X)}{\partial \theta}$, containing in the j-th row the derivative of F(θ, X_j) with respect to each of the parameters, $G_j(\theta)$ = $\frac{\partial F(\theta, X_j)}{\partial \theta}$:

$$\text{(2.4)} \qquad C\hat{o}v(\hat{\theta}_{MDMOS}) = [G(\hat{\theta})' \, G(\hat{\theta})]^{-1} \, G(\hat{\theta})' \, W \, G(\hat{\theta}) \, [G(\hat{\theta})' \, G(\hat{\theta})]^{-1}$$

[14] See Greene (2003), ps. 543-544, for example.

[15] See Martins (2005).

[16] See Greene (2003), for example, p. 544. Hansen (1982) establishes large sample properties of GMM estimators.

In general, for a truncated version of a "standard" (or fixed, specific) cdf F(.), i.e., F(θ, X_j)

$$= \frac{F(\frac{Y_j - a_1 - a_2 X_j}{\sigma}) - \alpha}{\beta - \alpha}$$

, $G_j(a_1, a_2, \sigma)$ can be computed from.

$$\text{(2.5)} \qquad G_j(a_1,a_2,\sigma) = \left[-\frac{f(\frac{Y_j - a_1 - a_2 X_j}{\sigma})}{\sigma(\beta-\alpha)} \; ; \; -\frac{f(\frac{Y_j - a_1 - a_2 X_j}{\sigma}) X_j}{\sigma(\beta-\alpha)} \; ; \; -\frac{f(\frac{Y_j - a_1 - a_2 X_j}{\sigma}) \frac{Y_j - a_1 - a_2 X_j}{\sigma^2}}{(\beta-\alpha)} \right]$$

f(.) denotes the density function associated to the (untruncated) cdf F(.).

We expect that for any consistent estimator of θ, $\hat{\theta}$, the identifying restriction test that relies on

$$\text{(2.6)} \qquad [S - F(\theta, X)]' W^{-1} [S - F(\theta, X)]$$

exhibits under the correct cdf an asymptotic distribution:

$$\text{(2.7)} \qquad [S - F(\hat{\theta}, X)]' W^{-1} [S - F(\hat{\theta}, X)] \sim X^2;_{(n-k)}$$ [17]

where n is the number of observations and k is the number of estimated parameters (at – upper tail - 5%, $X^2;_{(97)}$ = 120.9896; therefore, the order restrictions are not rejected at that significance level if the test statistic exhibits a lower value than the theoretical one).

. Minimum distance estimators for fixed α and β performed very well. Results are presented in Tables 2.1.1 to 2.1.5. Each Table contains the results of the simulation for a particular set of the endogenous variables, assuming three particular – normal, uniform and exponential - error term distribution as the null hypothesis – each column has information for a particular sample and truncated hypothesis. We compute, by nonlinear least squares, the parameter estimates, use formulas (2.4) and (2.5) to generate the appropriate standard error, and (2.7) the identifying restricted test statistic. e'e are the reported some of square errors reported by the routine.

We note that for the standard case (α = 0, β = 1, first row block for first column series), the s.e. are smaller than those obtained for OLS, reported in Tables 1.1.1-1.1.3, which points to the quality of the MDMOS method – however, the latter requires knowledge of the adequate cdf, being fully parametric in spirit.

[17] Which should also be useful as a rank test of a particular cdf form…

From the restriction test, we conclude that the true normals are identified by the results of Table 2.1.1. That uniform rather than the other alternatives applies to the error structure is patent in Table 2.1.2; but the method does not allow us to identify which truncated case. The exponential exhibits high test statistics but the true truncation appears with the best statistic value – Table 2.1.3. The logistic approaches a significance similar to the normal – Table 2.1.5. Cauchy disturbances lead to too large values of the restriction test statistic.

Table 2.1.1 Normal Disturbances

Null CDF	$\Phi(z)$	$\Phi(z)$	$\Phi(z)$	$\Phi[z\mid z<0]$	$\Phi[z\mid z<0$	$\Phi[z\mid z<0$	$\Phi[z\mid z>0]$	$\Phi[z\mid z>0]$	$\Phi[z\mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	4.92071 (0.11472)	6.34220 (0.099797)	3.49491 (0.096154)	4.33350 (0.070575)	5.07727 (0.060965)	3.35201 (0.063603)	5.56748 (0.069421)	6.53423 (0.064537)	4.85314 (0.058477)
$\hat{a}_2$	6.00484 (0.0023327)	6.01922 (0.0037286)	5.99189 (0.0034797)	5.98521 (0.0017899)	5.99172 (0.0025301)	5.97537 (0.0026361)	6.02099 (0.0017156)	6.03105 (0.0027087)	6.01529 (0.0024545)
$\hat{\sigma}$	1.05398 (0.086998)	2.06662 (0.14630)	2.07892 (0.14504)	.632329 (0.052349)	1.08075 (0.089439)	1.41621 (0.091364)	.619096 (0.052652)	1.38003 (0.090038)	1.03641 (0.088294)
e'e	.020518	.283056	0.35927	.108669	.022080	0.87523	.164915	.766555	.021202
$e'W^{-1}e$	66.68562	1059.65891	2860.84876	310.97348	106.38099	5413.88454	288.43845	2644.75254	103.62790
R^2	.997553	.966545	.957237	.987237	.997390	.894666	.980751	.907921	.997412
$\overline{R}^2$	.997503	.965855	.956356	.986974	.997336	.892494	.980354	.906022	.997359
Uniform									
$\hat{a}_1$	4.91914 (0.12892)	6.61296 (0.14954)	3.22532 (0.14633)	4.32355 (0.083329)	5.38070 (0.10524)	3.26639 (0.084793)	5.58022 (0.080552)	6.59619 (0.083755)	4.56424 (0.10073)
$\hat{a}_2$	6.00559 (0.0051120)	6.00559 (0.0051120)	6.00559 (0.0051120)	5.97995 (0.0035984)	5.97995 (0.0035984)	5.97995 (0.0035984)	6.02680 (0.0034997)	6.02680 (0.0034997)	6.02680 (0.0034997)
$\hat{\sigma}$	.977930 (0.041904)	1.95586 (0.083808)	1.95586 (0.083808)	.610347 (0.027010)	1.22069 (0.054020)	1.22069 (0.054020)	.586574 (0.026408)	1.17315 (0.052816)	1.17315 (0.052816)
e'e	.157005	.157005	.157005	.430546	.430546	.430546	.337181	.337181	.337181
$e'W^{-1}e$	1577.02995	1577.02996	1577.02996	3831.15616	3831.15616	3831.15616	1544.13307	1544.13307	1544.13307
R^2	.980779	.980779	.980779	.947291	.947291	.947291	.958721	.958721	.958721
$\overline{R}^2$	.980382	.980382	.980382	.946204	.946204	.946204	.957870	.957870	.957870
Exponential									
$\hat{a}_1$	5.56549 (0.19933)	8.64860 (0.34997)	4.12928 (0.078198)	4.83850 (0.14687)	6.88006 (0.25184)	3.75105 (0.058748)	5.80067 (0.092606)	7.56404 (0.18269)	5.21787 (0.036119)
$\hat{a}_2$	5.97999 (0.0038268)	5.99407 (0.0061963)	5.97999 (0.0028154)	5.96411 (0.0029833)	5.97297 (0.0044695)	5.96411 (0.0021961)	6.00978 (0.0023750)	6.02063 (0.0042687)	6.00979 (0.0017475)
$\hat{\sigma}$	2.07204 (0.15697)	5.24234 (0.32281)	2.07194 (0.11548)	1.56876 (0.10917)	3.58567 (0.22319)	1.56921 (0.080363)	.840806 (0.085322)	2.80389 (0.18233)	.840821 (0.062777)
e'e	1.07163	.317670	1.07163	1.91165	.896190	1.91165	.090777	.110217	.090777
$e'W^{-1}e$	6846.91313	3238.19579	6848.04295	10011.22109	6812.14420	10001.43335	457.76926	524.28622	457.71159
R^2	.880006	.961392	.879997	.781223	.891511	.781291	.989905	.986612	.989905
$\overline{R}^2$	.877532	.960596	.877523	.776713	.889274	.776782	.989696	.986336	.989697

Table 2.1.2 Uniform Disturbances

Null CDF	$U(z)$	$U(z)$	$U(z)$	$U[z \mid z<0]$	$U[z \mid z<0]$	$U[z \mid z<0]$	$U[z \mid z>0]$	$U[z \mid z>0]$	$U[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	4.91896 (0.13750)	6.49788 (0.11068)	3.40167 (0.10526)	4.09346 (0.068751)	4.88291 (0.055341)	3.33481 (0.052628)	5.82551 (0.068751)	6.61497 (0.055341)	5.06686 (0.052628)
$\hat{a}_2$	6.00358 (0.0032897)	6.01222 (0.0041707)	5.99522 (0.0040048)	6.00179 (0.0016448)	6.00611 (0.0020854)	5.99761 (0.0020024)	6.00179 (0.0016448)	6.00611 (0.0020854)	5.99761 (0.0020024)
$\hat{\sigma}$	1.21453 (0.092321)	2.20822 (0.16043)	2.13295 (0.15933)	.607265 (0.046161)	1.10411 (0.080215)	1.06648 (0.079667)	.607265 (0.046161)	1.10411 (0.080215)	1.06648 (0.079667)
e'e	.103576	.218054	.132322	.103576	.218054	.132322	.103576	.218055	0.13232
e'W^{-1}e	167.92384	356.39565	360.96625	167.92541	356.39314	360.96706	167.92389	356.39293	360.96671
R^2	.987623	.974152	.984378	.987623	.974152	.984378	.987623	.974152	.984378
$\overline{R}^2$	.987367	.973619	.984056	.987367	.973619	.984056	.987367	.973619	.984056
Uniform									
$\hat{a}_1$	4.92472 (0.13479)	6.69939 (0.15842)	3.15006 (0.15818)	4.09634 (0.067395)	4.98367 (0.079212)	3.20900 (0.079090	5.82839 (0.067395)	6.71572 (0.079212)	4.94105 (0.079090)
$\hat{a}_2$	6.00455 (0.0053222)	6.00455 (0.0053222	6.00455 (0.0053222	6.00228 (0.0026611)	6.00228 (0.0026611)	6.00228 (0.0026611)	6.00228 (0.0026611)	6.00228 (0.0026611)	6.00228 (0.0026611)
$\hat{\sigma}$	1.02460 (0.047928)	2.04921 (0.095856)	2.04921 (0.095856)	.512302 (0.023964)	1.02460 (0.047928)	1.02460 (0.047928)	.512302 (0.023964)	1.02460 (0.047928)	1.02460 (0.047928)
e'e	.029263	.029263	.029263	.029263	.029263	.029263	.029264	.029264	.029264
e'W^{-1}e	65.85107	65.85107	65.85107	65.85319	65.85319	65.85319	65.85069	65.85069	65.85069
R^2	.996417	.996417	.996417	.996417	.996417	.996417	.996417	.996417	.996417
$\overline{R}^2$	.996344	.996344	.996344	.996344	.996344	.996344	.996344	.996344	.996344
Exponential									
$\hat{a}_1$	5.32339 (0.19019)	8.63876 (0.36462)	4.07955 (0.069507)	4.29567 (0.095094)	5.95336 (0.18231)	3.67375 (0.034753)	6.02772 (0.095094)	7.68541 (0.18231)	5.40580 (0.034753)
$\hat{a}_2$	5.99053 (0.0034909)	5.99808 (0.0063896)	5.99053 (0.0025685)	5.99527 (0.0017455)	5.99904 (0.0031948)	5.99526 (0.0012842)	5.99527 (0.0017455)	5.99904 (0.0031948)	5.99527 (0.0012842)
$\hat{\sigma}$	1.79449 (0.16739)	5.28507 (0.35497)	1.79448 (0.12316)	.897244 (0.083693)	2.64254 (0.17748)	.897241 (0.061577)	.897244 (0.083693)	2.64253 (0.17748)	.897241 (0.061578)
e'e	.331807	.043758	.331807	.331807	.043758	.331807	.331807	.043758	.331807
e'W^{-1}e	871.48657	183.36527	871.50585	871.48110	183.36732	871.50030	871.48628	183.36544	871.50555
R^2	.961701	.994657	.961701	.961701	.994657	.961701	.961701	.994657	.961701
$\overline{R}^2$	.960912	.994547	.960912	.960912	.994547	.960912	.960912	.994547	.960912

Table 2.1.3 Exponential Disturbances

Null CDF	$Exp(z)$	$Exp(z)$	$Exp(z)$	$Exp[z \mid z<0]$	$Exp[z \mid z<0]$	$Exp[z \mid z<0]$	$Exp[z \mid z>0]$	$Exp[z \mid z>0]$	$Exp[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	4.47461 (0.095100)	6.00192 (0.10467)	3.55461 (0.086461)	4.34424 (0.038040)	4.81605 (0.030974)	3.95256 (0.030580)	5.47461 (0.095100)	7.00192 (0.10467)	4.55461 (0.086461)
$\hat{a}_2$	6.04830 (0.0031791)	6.07337 (0.0052514)	6.03876 (0.0040914)	6.00672 (0.00097694)	6.00953 (0.0012028)	6.00439 (0.0012868)	6.04830 (0.0031791)	6.07337 (0.0052514)	6.03876 (0.0040914)
$\hat{\sigma}$	.838859 (0.078254)	2.24066 (0.14053)	1.36478 (0.12984)	.332841 (0.026348)	.656212 (0.045137)	.556293 (0.044895)	.838859 (0.078254)	2.24066 (0.14053)	1.36478 (0.12984)
e'e	.308525	1.31444	0.075166	.157765	.454278	.052449	.308526	1.31444	.075166
$e'W^{-1}e$	574.22108	5304.58829	151.95137	278.94471	1029.56966	140.30715	574.22355	5304.59006	151.95251
R^2	.964953	.841012	.991259	.981201	.945674	.993711	.964953	.841012	.991259
$\bar{R}^2$	.964230	.837734	.991079	.980813	.944554	.993582	.964230	.837734	.991079
Uniform									
$\hat{a}_1$	4.47309 (0.13337)	6.07009 (0.12840)	2.87608 (0.17134)	4.35229 (0.038770)	4.85276 (0.042356)	3.85182 (0.047974)	5.47309 (0.13337)	7.07009 (0.12840)	3.87608 (0.17134)
$\hat{a}_2$	6.06733 (0.0065579)	6.06733 (0.0065579)	6.06733 (0.0065579)	6.00788 (0.0016055)	6.00788 (0.0016055)	6.00788 (0.0016055)	6.06733 (0.0065579)	6.06733 (0.0065579)	6.06733 (0.0065579)
$\hat{\sigma}$	.922032 (0.041371)	1.84406 (0.082742)	1.84406 (0.082742)	.288947 (0.013474)	.577894 (0.026948)	.577894 (0.026948)	.922032 (0.041371)	1.84406 (0.082742)	1.84406 (0.082742)
e'e	.807630	.807630	.807630	.137606	.137606	.137606	.807630	.807630	.807630
$e'W^{-1}e$	3952.06264	3952.06268	3952.06264	414.90500	414.90500	414.90500	3952.06499	3952.06499	3952.06499
R^2	.901127	.901127	.901127	.983154	.983154	.983154	.901127	.901127	.901127
$\bar{R}^2$	.899088	.899088	.899088	.982806	.982806	.982806	.899088	.899088	.899088
Exponential									
$\hat{a}_1$	4.81797 (0.12234)	7.42116 (0.26113)	4.07201 (0.049954)	4.45384 (0.049418)	5.36118 (0.095719)	4.14064 (0.019012)	5.81797 (0.12234)	8.42116 (0.26113)	5.07201 (0.049954)
$\hat{a}_2$	6.02811 (0.0038926)	6.05473 (0.0077871)	6.02811 (0.0028640)	6.00232 (0.0011824)	6.00582 (0.0020266)	6.00232 (0.00086998)	6.02811 (0.0038926)	6.05473 (0.0077871)	6.02811 (0.0028640)
$\hat{\sigma}$	1.07620 (0.11826)	4.13902 (0.26606)	1.07620 (0.087007)	.451862 (0.044712)	1.43489 (0.095739)	.451862 (0.032897)	1.07620 (0.11826)	4.13902 (0.26606)	1.07620 (0.087007)
e'e	.045412	.393581	.045412	.147361	.038007	.147361	.045412	.393581	.045412
$e'W^{-1}e$	345.02232	1783.39731	345.02245	454.65328	122.27628	454.65207	345.02176	1783.39977	345.02189
R^2	.994556	.952323	.994556	.983207	.995358	.983207	.994556	.952323	.994556
$\bar{R}^2$	.994443	.951340	.994443	.982861	.995262	.982861	.994443	.951340	.994443

Table 2.1.4 Cauchy Disturbances

Null CDF	$\Phi(z)$	$C(z)$	$C(z)$	$C[z \mid z<0$	$C[z \mid z<0$	$C[z \mid z<0$	$C[z \mid z>0]$	$C[z \mid z>0]$	$C[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	5.51902 (0.13825)	9.12168 (0.23081)	3.03519 (0.17660)	6.05875 (0.19476)	8.15564 (0.20258)	2.94251 (0.28675)	4.72851 (0.16261)	8.45910 (0.37491)	3.31429 (0.16546)
$\hat{a}_2$	5.90617 (0.0023273)	5.92761 (0.0084279)	5.87796 (0.0056940)	5.65974 (0.0071525)	5.71536 (0.0098026)	5.49829 (0.018033)	6.19018 (0.0096832)	6.41234 (0.026284)	6.15489 (0.0099266)
$\hat{\sigma}$	1.31054 (0.13090)	5.26487 (0.29321)	3.83684 (0.27091)	1.78347 (0.16694)	3.48285 (0.27202)	5.71388 (0.39519)	1.38166 (0.13285)	7.06587 (0.45291)	2.29732 (0.22316)
e’e	.086574	2.24428	2.41826	.069588	.415032	2.89883	.290690	3.34809	.181120
$e'W^{-1}e$	113.46156	4946.56853	15262.84015	143.70113	2662.61323	23737.97936	970.63082	23497.76625	520.15163
R^2	.991667	.728113	.751030	.992862	.949429	.713387	.968606	.605369	.979466
$\overline{R}^2$	.991495	.722507	.745897	.992715	.948386	.707477	.967958	.597232	.979043
Uniform									
$\hat{a}_1$	9.65044 (1.59835)	30.0832 (1.92580)	-10.7823 (1.44923)	34.3505 (5.00230)	84.9010 (6.56709)	-16.2000 (3.86172)	3.52216 (0.57390)	9.51914 (0.48580)	-2.47481 (0.74450)
$\hat{a}_2$	5.26093 (0.067400)	5.26093 (0.067400)	5.26093 (0.067400)	1.39975 (0.32667)	1.39975 (0.32667)	1.39975 (0.32667)	6.44382 (0.036843)	6.44382 (0.036843)	6.44382 (0.036843)
$\hat{\sigma}$	11.7969 (0.34145)	23.5937 (0.68290)	23.5937 (0.68290)	29.1853 (1.15422)	58.3707 (2.30845)	58.3707 (2.30845)	3.46236 (0.14806)	6.92471 (0.29613)	6.92471 (0.29613)
e’e	6.05568	6.05568	6.05568	5.05202	5.05202	5.05202	2.92962	2.92962	2.92962
$e'W^{-1}e$	22100.13847	22100.13847	22100.13847	33843.46111	33843.46111	33843.46111	19103.60892	19103.60892	19103.60892
R^2	.258637	.258637	.258637	.381510	.381510	.381510	.641343	.641343	.641343
$\overline{R}^2$	.243352	.243352	.243352	.368758	.368758	.368758	.633948	.633948	.633948
Exponential									
$\hat{a}_1$	30.9714 (4.21042)	81.2375 (5.92691)	-2.36287 (1.84192)	79.8958 (8.62706)	188.486 (15.44783)	18.3159 (3.63764)	5.60017 (0.20428)	11.8097 (0.64118)	4.32324 (0.093860)
$\hat{a}_2$	4.25293 (0.11947)	4.81603 (0.12774)	4.25285 (0.087907)	-1.79215 (0.33809)	-.205151 (0.47372)	-1.79199 (0.24875)	6.11938 (0.0090074)	6.29752 (0.029306)	6.11940 (0.0066277)
$\hat{\sigma}$	48.0906 (2.72670)	96.4392 (4.41682)	48.0937 (2.00644)	88.8407 (6.25080)	202.492 (12.09456)	88.8383 (4.59884)	1.84205 (0.20468)	10.6247 (0.65964)	1.84202 (0.15059)
e’e	6.80875	6.48309	6.80875	5.31651	5.16203	5.31651	.261993	1.62069	.261993
$e'W^{-1}e$	13933.36129	16801.98208	13931.74699	35938.60234	35190.47160	35939.22991	1242.77839	11835.60909	1242.80296
R^2	.169448	.207839	.169452	.349877	.368115	.349876	.967926	.809838	.967926
$\overline{R}^2$	.152324	.191506	.152327	.336472	.355086	.336472	.967265	.805917	.967265

Table 2.1.5 Logistic Disturbances

Null CDF	$L(z)$	$L(z)$	$L(z)$	$L[z\mid z<0]$	$L[z\mid z<0]$	$L[z\mid z<0]$	$L[z\mid z>0]$	$L[z\mid z>0]$	$L[z\mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	4.92679 (0.10496)	6.27315 (0.094539)	3.54058 (0.093344)	4.42522 (0.068421)	5.13102 (0.060627)	3.39232 (0.067132)	5.48404 (0.066379)	6.48328 (0.067785)	4.80691 (0.057061)
$\hat{a}_2$	6.00455 (0.0020545)	6.02008 (0.0035173)	5.99185 (0.0033441)	5.98014 (0.0018488)	5.98748 (0.0026051)	5.96786 (0.0029936)	6.02503 (0.0017145)	6.03817 (0.0029856)	6.01887 (0.0023888)
$\hat{\sigma}$	.975191 (0.082667)	1.98182 (0.13862)	2.03779 (0.13890)	.614430 (0.052656)	1.04030 (0.089396)	1.50924 (0.095879)	.597525 (0.052807)	1.44619 (0.092253)	.993407 (0.088312)
e’e	.014426	.357927	.571491	.138026	.013812	1.29543	.206159	1.01246	.028425
e’W^{-1}e	60.50093	1480.45796	5036.13127	383.37281	84.54572	8541.01025	280.24819	3591.14146	93.47231
R^2	.998251	.957463	.931500	.983994	.998313	.843916	.976285	.877954	.996560
$\overline{R}^2$	.998215	.956585	.930088	.983664	.998278	.840697	.975796	.875438	.996489
Uniform									
$\hat{a}_1$	4.92383 (0.12606)	6.58142 (0.14579)	3.26625 (0.14145)	4.42441 (0.090877)	5.55993 (0.11651)	3.28890 (0.088493)	5.49286 (0.084016)	6.53956 (0.085027)	4.44617 (0.10608)
$\hat{a}_2$	6.00490 (0.0049937)	6.00490 (0.0049937)	6.00490 (0.0049937)	5.97084 (0.0040862)	5.97084 (0.0040862)	5.97084 (0.0040862)	6.03365 (0.0037508)	6.03365 (0.0037508)	6.03365 (0.0037508)
$\hat{\sigma}$	.957009 (0.039749)	1.91402 (0.079497)	1.91402 (0.079497)	.655590 (0.028544)	1.31118 (0.057088)	1.31118 (0.057088)	.604307 (0.026971)	1.20861 (0.053942)	1.20861 (0.053942)
e’e	.326685	.326685	.326685	.828091	.828091	.828091	.534695	.534695	.534695
e’W^{-1}e	3279.47552	3279.47552	3279.47552	7280.06959	7280.06959	7280.06959	2357.88391	2357.88391	2357.88391
R^2	.960006	.960006	.960006	.898622	.898622	.898622	.934540	.934540	.934540
$\overline{R}^2$	.959181	.959181	.959181	.896531	.896531	.896531	.933191	.933191	.933191
Exponential									
$\hat{a}_1$	5.70234 (0.21048)	8.72624 (0.34782)	4.12036 (0.086572)	5.14325 (0.17745)	7.39933 (0.28805)	3.78798 (0.073407)	5.72042 (0.088692)	7.48505 (0.18041)	5.16621 (0.034519)
$\hat{a}_2$	5.97590 (0.0043272)	5.99195 (0.0062471)	5.97590 (0.0031825)	5.94597 (0.0039545)	5.96056 (0.0053906)	5.94597 (0.0029094)	6.01267 (0.0023230)	6.02628 (0.0044488)	6.01267 (0.0017091)
$\hat{\sigma}$	2.28244 (0.15512)	5.31950 (0.31291)	2.28188 (0.11408)	1.95526 (0.12458)	4.13539 (0.24691)	1.95519 (0.091655)	.799552 (0.083528)	2.80628 (0.18090)	.799547 (0.061456)
e’e	1.65300	.629733	1.65300	2.74895	1.53694	2.74895	.061314	.214357	.061314
e’W^{-1}e	10337.06379	6303.56091	10346.14083	12977.06941	11117.34422	12978.57956	443.46157	896.02596	443.48121
R^2	.815729	.923719	.815668	.684043	.814521	.684033	.993109	.974013	.993109
$\overline{R}^2$	.811929	.922146	.811867	.677529	.810697	.677519	.992967	.973477	.992967

The next Tables apply to the double error structures. The complexification of the error structure led to the disapearance of the truncated effects: Under all series, the smaller test statistic always points to an untruncated distribution. Notice that the variance of the extra noise is always one – where the mother untruncated distribution has also the same variance. It is possible that with a larger value for the variance of the latter would result in recogizable truncated effects.

Table 2.2.1 Normal Disturbances - Double

Null CDF	$\Phi(z)$	$\Phi(z)$	$\Phi(z)$	$\Phi[z \mid z<0]$	$\Phi[z \mid z<0$	$\Phi[z \mid z<0$	$\Phi[z \mid z>0]$	$\Phi[z \mid z>0]$	$\Phi[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\hat{a}_1$	4.70092 (0.15714)	6.77076 (0.12871)	2.45249 (0.12875)	4.01674 (0.12977)	5.83549 (0.099182)	2.11482 (0.10991)	5.41692 (0.12573)	7.21312 (0.10202)	3.65828 (0.10484)
$\hat{a}_2$	6.03768 (0.0040934)	6.03830 (0.0046489)	6.02533 (0.0039941)	6.02333 (0.0027609)	6.02074 (0.0026407)	6.00991 (0.0027258)	6.04664 (0.0025202)	6.04374 (0.0028460)	6.03178 (0.0024902)
$\hat{\sigma}$	1.43171 (0.12265)	2.88116 (0.20556)	3.18993 (0.20951)	1.20281 (0.10616)	2.50553 (0.17489)	2.72862 (0.17987)	1.18657 (0.10180)	2.46417 (0.17033)	2.54658 (0.17175)
e’e	.044009	.373766	7.60346	.023457	.400054	.659652	128.832	.469140	.518002
e’W^{-1}e	96.92388	1896.20982	1993.09945	78.53578	1172.99058	1984.76091	118.05929	2373.14513	1223.46022
R^2	.994715	.955341	.999821	.997254	.952321	.921231	.996979	.943988	.938303
$\overline{R}^2$	.994607	.954421	.999818	.997198	.951338	.919607	.996948	.942833	.937030

3. Inverse Order – Inter-Quantile - Regression

. In section 2, we established the required ordered principles to generate estimation strategies after a first-step OLS run. For the truncated normal, the order regression approach relies on the fact that:

$$(3.1) \qquad \frac{Rank(\frac{Y_i - a_1 - a_2 X_i}{\sigma})}{n+1} = \frac{Rank(Y_i - a_2 X_i)}{n+1} \approx \frac{F(\frac{Y_i - a_1 - a_2 X_i}{\sigma}) - \alpha}{\beta - \alpha}$$

Observation rankings, S_i, were inferred from the OLS errors. We can invert the approximation to obtain:

$$(3.2) \qquad Y_i \approx a_1 + a_2 X_i + \sigma \; F^{-1}[\alpha + S_i(\beta - \alpha)]$$

An inverse order approach would use those same rankings, to build, for given, fixed, α and β:

$$(3.3) \qquad \hat{E}_i \approx F^{-1}[\alpha + S_i(\beta - \alpha)] = F^{-1}\{F(a) + S_i[F(b) - F(a)]\}$$

and regress by OLS, Y_i on X_i and the "theoretical" error, $\hat{E}_i$:

$$(3.4) \qquad Y_i \approx a_1 + a_2 X_i + \sigma \; \hat{E}_i + v_i$$

The quality of the fit would guide us to the true distribution. The variance of the estimates have to be scaled relative to the OLS formula – the standard errors of the parameter estimates multiplied by the square root of *(*the variance of $\hat{E}_i$ times 99 times the OLS regression coeficient estimate squared, plus the sum of squares of the regression, and then the sum divided by 97*)*, rather than multiplied by the standard error of the OLS regression as directly reported by the software.

Also, emcompassing tests could easily be constructed relying on the inclusion of more than one transformation – say, use the inverse normal and the inverse exponential - in the right hand-side of the regression (3.4) performed.

At this stage we caution the reader to the fact that the expected value of Y_i has a bias relative to $a_1 + a_2 X_i$ that may not be well aproximated through the inverse structure (3.4) in the term $\sigma \; \hat{E}_i$. Yet assyntotically, (3.4) should result in an adequate framing.

Results depicted in Tables 3.1.1 to 3.1.5 exhibit similar patterns to the previous section: the truncated normal is well approximated, exhibiting the smaller sum of squares residuals e'e – that includes the variance accounted by the last term of the regression) - as v'v. Uniform disturbances are identified as such but not the truncated case. Exponential case is correctly identified. Cauchy and Logistic appear as normal cases.

Table 3.1 Normal Disturbances

Null CDF	$\Phi(z)$	$\Phi(z)$	$\Phi(z)$	$\Phi[z \mid z<0]$	$\Phi[z \mid z<0$	$\Phi[z \mid z<0$	$\Phi[z \mid z>0]$	$\Phi[z \mid z>0]$	$\Phi[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	4.93863 (.022235) [0.22071]	6.05100 (.059489) [0.25113]	3.81849 (.062321) [0.24993]	4.36026 (.027862) [0.13621]	5.02086 (.018010) [0.15657]	3.71304 (.058805) [0.14980]	5.57321 (.028486) [0.13122]	6.19444 (.058136) [0.14533]	4.91995 (.017265) [0.15080]
$\hat{a}_2$	6.00278 (.00287776) [0.028565]	6.02600 (.00680141) [0.028712]	5.98104 (.00717939) [0.028792]	5.97465 (.00360914) [0.017644]	5.99404 (.00201171) [0.017489]	5.96016 (.00710378) [0.018096]	6.02781 (.00368868) [0.016991]	6.04169 (.00694093) [0.017351]	6.01115 (.00193648) [0.016914]
$\hat{\sigma}$	.993154 (.010212) [0.10136]	1.61166 (.040007) [0.16889]	1.60861 (.042230) [0.16936]	.600764 (.012807) [0.062608]	1.00627 (.011833) [0.10287]	.946342 (.041785) [0.10644]	.578070 (.013089) [0.060293]	.908160 (.040828) [0.10206]	.973048 (.011391) [0.099491]
v'v	.933235	5.18541	5.76121	1.45438	.455962	5.47860	1.52512	5.27668	.422297
e'e	91.95008	92.40961	92.65558	34.75845	34.45909	35.55237	32.36050	32.97262	32.21740
R^2	.999978	.999877	.999863	.999965	.999989	.999869	.999964	.999875	.999990
$\overline{R}^2$	.999977	.999874	.999860	.999964	.999989	.999866	.999963	.999872	.999990
Uniform									
$\overline{\hat{E}}$	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603
$SD_{\hat{E}}$	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752
Min $\hat{E}_i$	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149
Max $\hat{E}_i$	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490
$\hat{a}_1$	4.91910 (.030596) [0.22069]	6.58037 (.038665) [0.27889]	3.25784 (.038632) [0.27865]	4.31949 (.031123) [0.13555]	5.32071 (.040369) [0.17582]	3.31828 (.038241) [0.16655]	5.58338 (.026633) [0.13107]	6.55724 (.032688) [0.16088]	4.60952 (.034579) [0.17018]
$\hat{a}_2$	6.00560 (.00395980) [0.028562]	6.00560 (.00395980) [0.028562]	6.00560 (.00395980) [0.028562]	5.98053 (.00402948) [0.017549]	5.98053 (.00402948) [0.017549]	5.98053 (.00402948) [0.017549]	6.02634 (.00344822) [0.016970]	6.02634 (.00344822) [0.016970]	6.02634 (.00344822) [0.016970]
$\hat{\sigma}$	.959133 (.013633) [0.098335]	1.91827 (.027266) [0.19667]	1.91827 (.027266) [0.19667]	.578051 (.013873) [0.060421]	1.15610 (.027746) [0.12084]	1.15610 (.027746) [0.12084]	.562259 (.011872) [0.058427]	1.12452 (.023744) [0.11685]	1.12452 (.023744) [0.11685]
v'v	1.76718	1.76718	1.76718	1.82282	1.82282	1.82282	1.33446	1.33446	1.33446
e'e	91.93908	91.93908	91.93908	34.57539	34.57539	34.57539	32.32201	32.32201	32.32201
R^2	.999958	.999958	.999958	.999956	.999956	.999956	.999968	.999968	.999968
$\overline{R}^2$	.999957	.999957	.999957	.999955	.999955	.999955	.999968	.999968	.999968

Exponential									
$\bar{\hat{E}}$	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773
$SD_{\hat{E}}$	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805
Min $\hat{E}_i$	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503
Max $\hat{E}_i$	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512
$\hat{a}_1$	5.19366 (.088746) [0.22469]	6.98445 (.051265) [0.30694]	4.23158 (.092958) [0.23536]	4.53994 (.073663) [0.14335]	5.55018 (.060738) [0.19519]	3.97982 (.073271) [0.14259]	5.76217 (.034244) [0.13148]	6.81961 (.018392) [0.17652]	5.16703 (.036678) [0.14082]
$\hat{a}_2$	5.96909 (.011492) [0.029096]	5.99331 (.00477998) [0.028619]	5.96909 (.011492) [0.029096]	5.95054 (.00957497) [0.018633]	5.97235 (.00550745) [0.017699]	5.95054 (.00957496) [0.018633]	6.00247 (.00443112) [0.017013]	6.01836 (.00176141) [0.016905]	6.00247 (.00443112) [0.017013]
$\hat{\sigma}$	.962082 (.042421) [0.10740]	3.39770 (.058500) [0.35026]	.962082 (.042421) [0.10740]	.560120 (.035345) [0.068783]	2.01439 (.067403) [0.21661]	.560120 (.035345) [0.068783]	.595137 (.016357) [0.062802]	2.02630 (.021557) [0.20689]	.595137 (.016357) [0.062802]
v’v	14.5871	2.56979	14.5871	9.59828	3.37466	9.59828	2.19781	.349585	2.19781
e’e	93.51043	92.12229	93.51043	36.34944	34.85197	36.34944	32.39836	32.20020	32.39836
R^2	.999653	.999939	.999653	.999770	.999919	.999770	.999948	.999992	.999948
$\bar{R}^2$	.999646	.999938	.999646	.999765	.999917	.999765	.999947	.999992	.999947

Table 3.2 Uniform Disturbances

Null CDF	$U(z)$	$U(z)$	$U(z)$	$U[z\|z<0]$	$U[z\|z<0]$	$U[z\|z<0]$	$U[z\|z>0]$	$U[z\|z>0]$	$U[z\|z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	4.93886 (.050554) [0.23308]	6.09440 (.081349) [0.26535]	3.76141 (.071897) [0.26404]	4.10340 (.025277) [0.11654]	4.68117 (.040675) [0.13267]	3.51468 (.035949) [0.13202]	5.83545 (.025277) [0.11654]	6.41323 (.040675) [0.13267]	5.24673 (.035948) [0.13202]
$\hat{a}_2$	6.00251 (00654284) [0.030166]	6.02582 (.00929210) [0.030309]	5.98004 (.00827638) [0.030395]	6.00126 (.00327142) [0.015083]	6.01291 (.00464605) [0.015155]	5.99002 (.00413820) [0.015197]	6.00126 (.00327142) [0.015083]	6.01291 (.00464605) [0.015155]	5.99002 (.00413819) [0.015197]
$\hat{\sigma}$	1.02907 (.023217) [0.10704]	1.66707 (.054657) [0.17828]	1.68756 (.048683) [0.17879]	.514537 (.011608) [0.053520]	.833537 (.027329) [0.089141]	.843780 (.024341) [0.089393]	.514537 (.011608) [0.053520]	.833537 (.027329) [0.089141]	.843780 (.024341) [0.089393]
v'v	4.82412	9.68162	7.65867	1.20603	2.42040	1.91467	1.20603	2.42041	1.91467
e'e	102.54380	103.00736	103.29222	25.63594	25.75183	25.82305	25.63593	25.75182	25.82304
R^2	.999885	.999770	.999818	.999971	.999942	.999954	.999971	.999942	.999954
$\overline{R}^2$	.999883	.999765	.999814	.999971	.999941	.999953	.999971	.999941	.999953
Uniform									
$\overline{\hat{E}}$	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603
$SD_{\hat{E}}$	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752
Min $\hat{E}_i$	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149
Max $\hat{E}_i$	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490
$\hat{a}_1$	4.92470 (.013949) [0.23306]	6.69301 (.017648) [0.29485]	3.15639 (.017594) [0.29394]	4.09632 (.00697476) [0.11653]	4.98048 (.00882392) [0.14742]	3.21217 (.879690) [0.14697]	5.82837 (.00697478) [0.11653]	6.71253 (.00882395) [0.14742]	4.94422 (.00879692) [0.14697]
$\hat{a}_2$	6.00456 (.00180536) [0.030163]	6.00456 (.00180536) [0.030163]	6.00456 (.00180536) [0.030163]	6.00228 (.000902683) [0.015081]	6.00228 (.000902683) [0.015081]	6.00228 (.000902683) [0.015081]	6.00228 (.000902686) [0.015081]	6.00228 (.000902683) [0.015081]	6.00228 (.000902683) [0.015081]
$\hat{\sigma}$	1.02093 (.00621568) [0.10385]	2.04187 (.012431) [0.20769]	2.04187 (.012431) [0.20769]	.510467 (.00310785) [0.051923]	1.02093 (.621570) [0.10385]	1.02093 (.00621570) [0.10385]	.510467 (.00310786) [0.051923]	1.02093 (.00621570) [0.10385]	1.02093 (.00621570) [0.10385]
v'v	.367331	.367331	.367331	.091833	.091833	.091833	.091834	.091834	.091834
e'e	102.53398	102.53398	102.53398	25.63348	25.63348	25.63348	25.63348	25.63348	25.63348
R^2	.999991	.999991	.999991	.999998	.999998	.999998	.999998	.999998	.999998
$\overline{R}^2$	.999991	.999991	.999991	.999998	.999998	.999998	.999998	.999998	.999998

Exponential									
$\overline{\hat{E}}$	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773
$SD_{\hat{E}}$	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805
Min $\hat{E}_i$	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503
Max $\hat{E}_i$	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512
$\hat{a}_1$	5.19568 (.101601) [0.23696]	7.12124 (.036904) [0.32446]	4.19739 (.106613) [0.24865]	4.23182 (.050800) [0.11848]	5.19459 (.018452) [0.16223]	3.73267 (.053307) [0.12432]	5.96387 (.050800) [0.11848]	6.92664 (.018452) [0.16223]	5.46472 (.053307) [0.12432]
$\hat{a}_2$	5.96868 (.013155) [0.030682]	5.99169 (.00343825) [0.030229]	5.96868 (.013155) [0.030682]	5.98434 (.00657764) [0.015341]	5.99585 (.00171913) [0.015114]	5.98434 (.00657764) [0.015341]	5.98434 (.00657764) [0.015341]	5.99585 (.00171913) [0.015114]	5.98434 (.00657763) [0.015341]
$\hat{\sigma}$	.998289 (.048561) [0.11326]	3.61597 (.042079) [0.36995]	.998290 (.048561) [0.11326]	.499144 (.024281) [0.056629]	1.80799 (.021040) [0.18498]	.499145 (.024281) [0.056629]	.499145 (.024281) [0.056629]	1.80798 (.021039) [0.18498]	.499145 (.024281) [0.056629]
v’v	19.1410	1.32937	19.1410	4.78525	.332345	4.78525	4.78524	.332343	4.78524
e’e	104.11651	102.75756	104.11651	26.02912	25.68938	26.02912	26.02911	25.68937	26.02911
R^2	.999544	.999968	.999544	.999886	.999992	.999886	.999886	.999992	.999886
$\overline{R}^2$	.999535	.999968	.999535	.999883	.999992	.999883	.999883	.999992	.999883

Table 3.3 Exponential Disturbances

Null CDF	$Exp(z)$	$Exp(z)$	$Exp(z)$	$Exp[z \mid z<0]$	$Exp[z \mid z<0]$	$Exp[z \mid z<0]$	$Exp[z \mid z>0]$	$Exp[z \mid z>0]$	$Exp[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	4.47073 (.056192) [0.20087]	5.37762 (.102391) [0.21800]	3.50426 (.025791) [0.23217]	4.34761 (.014932) [0.065360]	4.66546 (.026635) [0.073165]	4.02068 (.015850) [0.074887]	5.47073 (.056192) [0.20087]	6.37762 (.102391) [0.21800]	4.50426 (.025791) [0.23217]
$\hat{a}_2$	6.06767 (.00728464) [0.026041]	6.08949 (.012658) [0.026950]	6.03902 (.00284554) [0.025616]	6.00855 (.00193295) [0.0084607]	6.01501 (.00312300) [0.0085788]	6.00117 (.00179078) [0.0084609]	6.06767 (.00728464) [0.026041]	6.08949 (.012658) [0.026950]	6.03902 (.00284554) [0.025616]
$\hat{\sigma}$	.866222 (.025849) [0.092404]	1.33935 (.074457) [0.15852]	1.47471 (.016738) [0.15068]	.287569 (.00685895) [0.030022]	.458971 (.018370) [0.050462]	.478561 (.010534) [0.049769]	.866222 (.025849) [0.092404]	1.33935 (.074457) [0.15852]	1.47471 (.016738) [0.15068]
v'v	5.87822	17.0512	.912409	.420226	1.08068	.360677	5.87823	17.0512	.912410
e'e	75.11674	77.29008	73.94315	8.05109	8.15462	8.05138	75.11674	77.29008	73.94315
R^2	.999861	.999598	.999978	.999990	.999974	.999991	.999861	.999598	.999978
$\overline{R}^2$	.999859	.999590	.999978	.999990	.999974	.999991	.999859	.999590	.999978
Uniform									
$\overline{\hat{E}}$	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603
$SD_{\hat{E}}$	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752
Min $\hat{E}_i$	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149
Max $\hat{E}_i$	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490
$\hat{a}_1$	4.49507 (.063021) [0.20028]	5.93194 (.075448) [0.23978]	3.05819 (.083635) [0.26580]	4.35264 (.00847783) [0.065317]	4.84467 (.010542) [0.081222]	3.86061 (.010874) [0.083781]	5.49507 (.063021) [0.20028]	6.93194 (.075449) [0.23978]	4.05819 (.083635) [0.26580]
$\hat{a}_2$	6.06416 (.00816730) [0.025956]	6.06416 (.00816730) [0.025956]	6.06416 (.00816730) [0.025956]	6.00783 (.00109735) [0.0084545]	6.00783 (.00109735) [0.0084545]	6.00783 (.00109735) [0.0084545]	6.06416 (.00816730) [0.025956]	6.06416 (.816730) [0.025956]	6.06416 (.816730) [0.025956
$\hat{\sigma}$	.829579 (.028119) [0.089363]	1.65916 (.056238) [0.17873]	1.65916 (.056238) [0.17873]	.284073 (.00377806) [0.029108]	.568146 (.00755612) [0.058216]	.568146 (.00755612) [0.058216]	.829579 (.028119) [0.089363]	1.65916 (.056238) [0.17873]	1.65916 (.056238) [0.17873]
v'v	7.41306	7.41306	7.41306	.135540	.135540	.135540	7.41306	7.41306	7.41306
e'e	74.87048	74.87048	74.87047	8.04550	8.04550	8.04550	74.87048	74.87048	74.87048
R^2	.999825	.999825	.999825	.999997	.999997	.999997	.999825	.999825	.999825
$\overline{R}^2$	.999822	.999822	.999822	.999997	.999997	.999997	.999822	.999822	.999822

Exponential									
$\overline{\hat{E}}$	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773
$SD_{\hat{E}}$	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805
Min $\hat{E}_i$	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503
Max $\hat{E}_i$	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512
$\hat{a}_1$	4.80018 (.032597) [0.19846]	6.34397 (.057771) [0.26264]	3.88052 (.035328) [0.21508]	4.42917 (.024025) [0.065772]	4.97045 (.00564367) [0.089168]	4.14234 (.025639) [0.070193]	5.80018 (.032597) [0.19846]	7.34397 (.057771) [0.26264]	4.88052 (.035328) [0.21508]
$\hat{a}_2$	6.02311 (.00421746) [0.025677]	6.05110 (.00565521) [0.025710]	6.02311 (.00421746) [0.025677]	5.99771 (.00310896) [0.0085114]	6.00410 (.00534474) [0.0084445]	5.99771 (.00310896) [0.0085114]	6.02311 (.00421746) [0.025677]	6.05110 (.00565521) [0.025710]	6.02311 (.00421746) [0.025677]
$\hat{\sigma}$	.919653 (.015568) [0.094784]	3.01713 (.069211) [0.31465]	.919653 (.015568) [0.094784]	.286830 (.011476) [0.031419]	1.01577 (.00654115) [0.10335]	.286830 (.011476) [0.031419]	.919653 (.015568) [0.094784]	3.01713 (.069211) [0.31465]	.919653 (.015568) [0.094784]
v’v	1.99951	3.59036	1.99951	1.08007	.032192	1.08007	1.99951	3.59036	1.99951
e’e	74.11503	74.20545	74.11503	8.09511	8.03611	8.09511	74.11503	74.20545	74.11503
R^2	.999953	.999915	.999953	.999974	.999999	.999974	.999953	.999915	.999953
$\overline{R}^2$	.999952	.999914	.999952	.999974	.999999	.999974	.999952	.999914	.999952

Table 3.4 Cauchy Disturbances

Null CDF	$C(z)$	$C(z)$	$C(z)$	$C[z\mid z<0]$	$C[z\mid z<0$	$C[z\mid z<0$	$C[z\mid z>0]$	$C[z\mid z>0]$	$C[z\mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	6.33758 (1.06073) [1.29372]	10.3018 (1.24777) [1.54920]	2.86971 (1.19594) [1.37088]	10.6326 (2.31195) [2.76711]	17.4552 (2.84335) [3.45266]	3.92328 (2.20368) [2.47993]	4.33525 (.332085) [0.62364]	6.67858 (.427166) [0.63620]	1.73713 (.323621) [0.73732]
$\hat{a}_2$	5.73868 (.137725) [0.16798]	5.81085 (.133235) [0.16542]	5.70457 (.149394) [0.17125]	4.82018 (.306483) [0.36682]	5.01526 (.279291) [0.33914]	4.82005 (.351113) [0.39513]	6.32657 (.043299) [0.081313]	6.37774 (.058190) [0.086666]	6.24639 (.034025) [0.077522]
$\hat{\sigma}$	3.30473 (.488708) [0.59605]	5.65079 (.783708) [0.97303]	4.68865 (.878755) [1.00730]	6.13769 (1.08753) [1.30164]	10.3476 (1.64283) [1.99488]	8.49299 (2.06530) [2.32419]	2.31255 (.153643) [0.28854]	3.41509 (.342285) [0.50978]	3.99207 (.200141) [0.45599]
v’v	2067.02	1980.14	2351.35	8037.39	7577.42	9091.58	195.306	321.504	127.696
e’e	3074.78750	3052.42694	3089.56956	11513.54370	11172.99620	11513.79127	688.78961	713.15072	662.85936
R^2	.952589	.954582	.946067	.832609	.842189	.810654	.995617	.992785	.997134
$\overline{R}^2$	.951612	.953645	.944955	.829158	.838935	.806750	.995526	.992636	.997075
Uniform									
$\overline{\hat{E}}$	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603
$SD_{\hat{E}}$	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752
Min $\hat{E}_i$	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149
Max $\hat{E}_i$	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490
$\hat{a}_1$	6.20036 (1.12939) [1.29357]	10.9246 (1.54164) [1.76575]	1.47608 (1.30558) [1.49537]	9.81915 (2.44323) [2.76098]	18.4800 (3.75370) [4.24188]	1.15827 (2.30845) [2.60866]	4.32985 (.382035) [0.62693]	7.97116 (.416823) [0.68402]	.688535 (.543158) [0.89134]
$\hat{a}_2$	5.75847 (.146678) [0.16800]	5.75847 (.146678) [0.16800]	5.75847 (.146678) [0.16800]	4.93748 (.324312) [0.36649]	4.93748 (.324312) [0.36649]	4.93748 (.324312) [0.36649]	6.32734 (.049867) [0.081833	6.32734 (.049867) [0.081833]	6.32734 (.049867) [0.081833]
$\hat{\sigma}$	2.72756 (.504997) [0.57841]	5.45512 (1.00999) [1.15682]	5.45512 (1.00999) [1.15682]	5.00036 (1.11657) [1.26179]	10.0007 (2.23315) [2.52357]	10.0007 (2.23315) [2.52357]	2.10231 (.171688) [0.28174]	4.20462 (.343375) [0.56349]	4.20462 (.343375) [0.56349]
v’v	2338.22	2338.22	2338.22	8847.33	8847.33	8847.33	255.895	255.895	255.895
e’e	3067.45134	3067.45134	3067.45135	11298.17517	11298.17521	11298.17510	689.11482	689.11482	689.11481
R^2	.946368	.946368	.946368	.815741	.815741	.815741	.994257	.994257	.994257
$\overline{R}^2$	.945263	.945263	.945263	.811942	.811942	.811942	.994139	.994139	.994139

Exponential									
$\overline{\hat{E}}$	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773
$SD_{\hat{E}}$	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805
Min $\hat{E}_i$	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503
Max $\hat{E}_i$	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512
$\hat{a}_1$	6.73091 (1.17406) [1.32884]	11.9592 (1.71500) [1.95397]	3.90839 (1.15891) [1.31170]	10.5332 (2.75182) [3.03486]	20.3445 (4.29884) [4.82903]	5.64539 (2.24753) [2.47870]	5.39120 (.192972) [0.58845]	9.08457 (.419374) [0.74377]	2.77706 (.216720) [0.66086]
$\hat{a}_2$	5.69103 (.152725) [0.17286]	5.73027 (.148977) [0.16974]	5.69103 (.152725) [0.17286]	4.85021 (.367820) [0.40565]	4.88656 (.338440) [0.38018]	4.85021 (.367820) [0.40565]	6.18268 (.024991) [0.076208]	6.28903 (.044780) [0.079418]	6.18268 (.024991) [0.076208]
$\hat{\sigma}$	2.82252 (.563770) [0.63810]	9.54595 (1.82325) [2.07730]	2.82252 (.563770) [0.63810]	4.88779 (1.35777) [1.49743]	17.4544 (4.14199) [4.65283]	4.88779 (1.35777) [1.49743]	2.61414 (.092253) [0.28131]	7.70277 (.548034) [0.97196]	2.61414 (.092253) [0.28131]
v’v	2416.90	2371.30	2416.90	9418.29	9024.43	9418.29	70.2144	214.532	70.2146
e’e	3096.18891	3078.18376	3096.18897	11455.36441	11387.74032	11455.36415	652.90423	674.79155	652.90423
R^2	.944564	.945610	.944564	.803850	.812052	.803850	.998424	.995185	.998424
$\overline{R}^2$	.943421	.944488	.943421	.799805	.808177	.799805	.998392	.995086	.998392

Table 3.5 Logistic Disturbances

Null CDF	$L(z)$	$L(z)$	$L(z)$	$L[z \mid z<0]$	$L[z \mid z<0]$	$L[z \mid z<0]$	$L[z \mid z>0]$	$L[z \mid z>0]$	$L[z \mid z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	4.94335 (.023096) [0.21369]	6.02160 (.056933) [0.24322]	3.86470 (.065875) [0.24195]	4.46230 (.037077) [0.14296]	5.14514 (.023529) [0.16493]	3.80056 (.069811) [0.15569]	5.48886 (.032136) [0.13359]	6.10707 (.063536) [0.14705]	4.83041 (.013558) [0.15370]
$\hat{a}_2$	6.00209 (.00298917) [0.027657]	6.02449 (.00650613) [0.027795]	5.98133 (.00758964) [0.027876]	5.96537 (.00480580) [0.018530]	5.98615 (.00260406) [0.018254]	5.95095 (.00855934) [0.019088]	6.03423 (.00416239) [0.017303]	6.04880 (.00766823) [0.017747]	6.01629 (.00151498) [0.017174]
$\hat{\sigma}$	.960823 (.010607) [0.098138]	1.56137 (.038270) [0.16349]	1.54743 (.044643) [0.16397]	.620570 (.017053) [0.065751]	1.04666 (.015317) [0.10737]	.964100 (.050347) [0.11228]	.584591 (.014770) [0.061400]	.910375 (.045106) [0.10439]	.990871 (.00891135) [0.10102]
v'v	1.00688	4.74570	6.43816	2.56278	.764230	7.85537	1.93688	6.38851	.258582
e'e	86.19428	86.61110	86.84915	38.09900	37.55166	39.06839	33.47192	34.21969	33.22913
R^2	.999976	.999887	.999847	.999939	.999982	.999812	.999954	.999848	.999994
$\overline{R}^2$	.999976	.999885	.999844	.999937	.999981	.999808	.999953	.999845	.999994
Uniform									
$\overline{\hat{E}}$	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603	8.19564D-09	-0.86603	0.86603
$SD_{\hat{E}}$	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752	0.99504	0.49752	0.49752
Min $\hat{E}_i$	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149	-1.69775	-1.71490	0.017149
Max $\hat{E}_i$	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490	1.69775	-0.017149	1.71490
$\hat{a}_1$	4.92373 (.042730) [0.21367]	6.51502 (.054008) [0.27006]	3.33243 (.053944) [0.26974]	4.41290 (.045204) [0.14192]	5.43245 (.059198) [0.18585]	3.39335 (.054957) [0.17254]	5.49936 (.034141) [0.13341]	6.47708 (.041592) [0.16253]	4.52164 (.044629) [0.17440]
$\hat{a}_2$	6.00492 (.00553020) [0.027653]	6.00492 (.00553020) [0.027653]	6.00492 (.00553020) [0.027653]	5.97250 (.00585454) [0.018381]	5.97250 (.00585454) [0.018381]	5.97250 (.00585454) [0.018381]	6.03271 (.00442142) [0.017278]	6.03271 (.00442142) [0.017278]	6.03271 (.00442142) [0.017278]
$\hat{\sigma}$	.918734 (.019040) [0.095207]	1.83747 (.038080) [0.19041]	1.83747 (.038080) [0.19041]	.588639 (.020157) [0.063282]	1.17728 (.040313) [0.12656]	1.17728 (.040313) [0.12656]	.564488 (.015223) [0.059485]	1.12898 (.030445) [0.017278]	1.12898 (.030445) [0.017278]
v'v	3.44680	3.44680	3.44680	3.83479	3.83479	3.83479	2.18877	2.18877	2.18877
e'e	86.18257	86.18257	86.18257	37.79830	37.79830	37.79830	33.42250	33.42250	33.42250
R^2	.999918	.999918	.999918	.999908	.999908	.999908	.999948	.999948	.999948

$\overline{R}^2$	.999916	.999916	.999916	.999906	.999906	.999906	.999947	.999947	.999947
Exponential									
$\overline{\hat{E}}$	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773	-0.022273	-0.58279	0.97773
$SD_{\hat{E}}$	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805	0.92805	0.27992	0.92805
Min $\hat{E}_i$	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503	-0.99005	-0.99372	0.0099503
Max $\hat{E}_i$	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512	3.61512	-0.016870	4.61512
$\hat{a}_1$	5.18814 (.087503) [0.21754]	6.90145 (.065685) [0.29721]	4.25991 (.091652) [0.22785]	4.64395 (.083815) [0.15165]	5.66287 (.080391) [0.20693]	4.07301 (.082212) [0.14875]	5.69501 (.027611) [0.13348]	6.74794 (.028285) [0.17824]	5.08256 (.029622) [0.14320]
$\hat{a}_2$	5.96977 (.011331) [0.028170]	5.99318 (.00612380) [0.027709]	5.96977 (.011331) [0.028170]	5.94101 (.010910) [0.019740]	5.96402 (.00721878) [0.018581]	5.94101 (.010910) [0.019740]	6.00647 (.00357268) [0.017271]	6.02444 (.00272678) [0.017183]	6.00647 (.00357268) [0.017271]
$\hat{\sigma}$	.928229 (.041827) [0.10398]	3.25394 (.074946) [0.33912]	.928230 (.041827) [0.10398]	.570936 (.040274) [0.072869]	2.04397 (.088347) [0.22741]	.570936 (.040274) [0.072869]	.612450 (.013188) [0.063755]	2.04400 (.033372) [0.21030]	.612450 (.013188) [0.063755]
v'v	14.1811	4.21770	14.1811	12.2241	5.76092	12.2241	1.42976	.837232	1.42975
e'e	87.64799	86.35256	87.64799	40.01843	38.16931	40.01843	33.41300	33.24675	33.41300
R^2	.999662	.999900	.999662	.999707	.999862	.999707	.999966	.999980	.999966
$\overline{R}^2$	.999655	.999897	.999655	.999701	.999859	.999701	.999965	.999980	.999965

In the next Table, we repeat the procedure for normal disturbances and hypothesis with the double error structure. Again the “best” results fall on the untruncated cases for each of the series.

Table 3.2.1 Normal Disturbances

Null CDF	$\Phi(z)$	$\Phi(z)$	$\Phi(z)$	$\Phi[z\,\vert\,z<0]$	$\Phi[z\,\vert\,z<0$	$\Phi[z\,\vert\,z<0$	$\Phi[z\,\vert\,z>0]$	$\Phi[z\,\vert\,z>0]$	$\Phi[z\,\vert\,z>0]$
Normal	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞	a= -∞; b= ∞	a= -∞; b= 0	a= 0; b= ∞
$\overline{\hat{E}}$	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008	1.32620D-08	-0.79008	0.79008
$SD_{\hat{E}}$	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241	0.96544	0.58241	0.58241
Min $\hat{E}_i$	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409	-2.33008	-2.57927	0.012409
Max $\hat{E}_i$	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927	2.33008	-0.012409	2.57927
$\hat{a}_1$	4.67505 (.032487) [0.31716]	6.41873 (.073411) [0.36690]	2.99396 (.113943) [0.36314]	3.99149 (.020901) [0.27221]	5.51652 (.065118) [0.31602]	2.52217 (.094114) [0.31301]	5.41093 (.020923) [0.26128]	6.84957 (.072795) [0.30318]	4.00142 (.081211) [0.29931]
$\hat{a}_2$	6.03445 (.00420462) [0.041048]	6.04848 (.821578) [0.041062]	6.01934 (.012912) [0.041149]	6.02149 (.00270511) [0.035230]	6.02912 (.00725986) [0.035233]	6.01146 (.010608) [0.035281]	6.04487 (.00270787) [0.033816]	6.05406 (.811947) [0.033817]	6.03308 (.00919646) [0.033895]
$\hat{\sigma}$	1.42687 (.014920) [0.14565]	2.33010 (.048326) [0.24153]	2.26035 (.075948) [0.24205]	1.22753 (.00959891) [0.12501]	1.99715 (.042704) [0.20725]	1.94776 (.062398) [0.20753]	1.17791 (.960870) [0.12000]	1.90157 (.047760) [0.19891]	1.88749 (.054095) [0.19937]
v’v	1.99214	7.60346	18.7365	.824647	5.93905	12.6623	.826241	7.42851	9.50699
e’e	189.86327	189.92589	190.30736	139.86901	139.88034	140.05994	128.85599	128.85619	129.14321
R^2	.999953	.999821	.999560	.999981	.999860	.999701	.999981	.999826	.999777
$\overline{R}^2$	.999952	.999818	.999551	.999980	.999857	.999695	.999981	.999822	.999772

4. Moment Replication

The deterministic part of a linear model is added of an error which is range restricted, ε_i, and a conventional one, v_{0i}. We can unfold k general equation blocks to the model by multiplying for each block each equation by the observations of one of the k + 1 (including the constant term) explanatory variables:

$$(4.1) \qquad \begin{aligned} Y_i &= \beta_0 + X_{1i}\beta_1 + X_{2i}\beta_2 + \dots + X_{ki}\beta_k + \varepsilon_i + v_{0i} \\ X_{1i}Y_i &= X_{1i}\beta_0 + X_{1i}^2\beta_1 + X_{1i}X_{2i}\beta_2 + \dots + X_{1i}X_{ki}\beta_k + X_{2i}\varepsilon_i + v_{1i} \\ X_{2i}Y_i &= X_{2i}\beta_0 + X_{2i}X_{1i}\beta_1 + X_{2i}^2\beta_2 + \dots + X_{2i}X_{ki}\beta_k + X_{2i}\varepsilon_i + v_{2i} \\ &\dots \\ X_{ki}Y_i &= X_{ki}\beta_0 + X_{ki}X_{1i}\beta_1 + X_{ki}X_{2i}\beta_2 + \dots + X_{ki}^2\beta_k + X_{ki}\varepsilon_i + v_{ki} \\ & \qquad i = 1,2,\dots,n \end{aligned}$$

We want to restrict $\varepsilon_i > 0$, or $\varepsilon_i < 0$, to denote the effective inefficiency. Then we could hope to parametrize our simple linear regression as a nonlinear one with 2n observations [18]:

$$(4.2) \qquad \begin{aligned} Y_i &= 1\ a_1 + X_i\, a_2 + 1\sqrt{\varepsilon_i^2} + v_{0i}, \quad i = 1,2,\dots,n \\ X_iY_i &= X_i\, a_1 + X_i^2\, a_2 + X_i\sqrt{\varepsilon_i^2} + v_{1i}, \quad i = 1,2,\dots,n \end{aligned}$$

$\sqrt{\varepsilon_i^2}$ would be n parameter to be estimated along with a and b. The variable that multiplies each of the $\sqrt{\varepsilon_i^2}$'s has zeros all over with the exception of the ith observation – 1 – and the n+i-th observation – that contains X_i. The previous case contemplates minimum cost structures; for a production or revenue frontier we can use:

$$(4.3) \qquad \begin{aligned} Y_i &= 1\ a + X_i\, b - 1\sqrt{\varepsilon_i^2} + v_{0i}, \quad i = 1,2,\dots,n \\ X_iY_i &= X_i\, a + X_i^2\, b - X_i\sqrt{\varepsilon_i^2} + v_{1i}, \quad i = 1,2,\dots,n \end{aligned}$$

To account for potential heteroscedasticity, the data of the second equation block was divided by the square root of the mean of X_i^2 [19] in a weighted least squares – WLS - version of the double equation model. We then considered a covariance matrix of the two equation system typical

[18] Method accuracy should increase with k – and the number of observations.

[19] Alternativelly, restricted SUR could be preformed on the system of two equations. The covariance structure would then allow for the mean of X to factor the covariance between the same observations of the two blocks…

error that off of he diagonal has the mean of X divided be the square root of the mean of X_i^2 and applied GLS in accordance to the previous system (WLS) – a GLS refinement of the procedure.

The first comment one can make is that even if $v_{ki} = 0$, least squares procedures cannot solve for the k + 1 parameters and the ε_i's – nor the $\sqrt{\varepsilon_i^2}$ - directly as parameters of an equation system – we attain singularity. Therefore we present estimates of the model

$$(4.4) \qquad Y_i = 1\ a + X_i\ b + X_i\ b + \sigma\ \hat{E}_i + v_{0i},\ \ i = 1,2,\ldots,n$$

$$X_i Y_i = X_i\ a + X_i^2\ b + \sigma\ \hat{E}_i\ X_i + v_{1i},\ i = 1,2,\ldots,n$$

where

$$(4.5) \qquad \hat{E}_i \approx F^{-1}[\alpha + S_i(\beta - \alpha)]$$

and S_i the frequency estimated from the rankings of the first step OLS regression. I.e., we apply the method of replicated moments to the first equation of form (4.4).

We also present the results for the standard linear regression model – (1.10). We experimented adding the 98 quasi-dummies – we discarded the two (true) median observations – to the simple linear regression. We discard additionally the last observation error when $\hat{E}_i$ is included in the regression.

We restricted the results to the normal case. Table 4.1 refers to single error structure. The correct truncation is invariably identified.

The second table used the the double-error series. As in previous sections, the correct truncation hypothesis does not exhibit the smaller sum of squares, classifying incorrectly the true truncation case. The added error dummies originated nonsensical parameters estimates, even for a_2, the slope coefficient, that is always well captured by the different procedures.

Table 4.1 Single Normal Disturbances												
				Null CDF								
				$\Phi(z)$			$\Phi[z \mid z<0]$			$\Phi[z \mid z>0]$		
Sample	WLS	With errors	GLS	WLS	With errors	GLS	WLS	With errors	GLS	WLS	With errors	GLS
a = ∞; b = ∞												
$\hat{a}_1$	4.72733 (0.186440)	4.97580	4.83980 (.123789)	4.94910 (.018958)	4.91321	4.93966 (.012405)	6.01180 (.047691)	6.78900	6.05974 (.033212	3.86216 (.049124)	4.24159	3.80950 (.033339
$\hat{a}_2$	6.03066 (.020619)	5.99708	6.02857 ().014313	6.00140 (.00209977)	6.00767	6.00191 (.00144451)	6.03090 (.0047121)	6.02037	6.02819 (.00312004	5.97584 (.00523983)	6.00518	5.97962 (.00339527
$\overline{\varepsilon}$		0.00024477*		.983153 (.00711541	.830074	.980795 (.00700813)	1.60854 (.026831)	2.89198	1.61608 (.025649	1.60224 (.029323)	1.01743	1.60197 (.026921
SD_{ε}		0.97404*										
$\hat{\sigma}$	1.03739		1.00426	.105105		.100468	.237076		.218915	.258748		.231130
R^2	.999032		.999514	.999990		.999995	.999950		.999977	.999940		.999974
$\overline{R}^2$	.999027		.999511	.999990		.999995	.999949		.999977	.999939		.999974
a = ∞; b =												
$\hat{a}_1$	4.15288 (.113666)	4.31028	4.19670 (.075096)	4.41444 (.025357	4.32587	4.36359 (.017011)	5.00911 (.015284)	5.04865	5.01536 (.011067	3.83039 (.047086)	4.02169	3.76901 (.032411)
$\hat{a}_2$	6.00321 (.012571)	5.99813	.600352 (.00868286)	5.96757 (.00282407)	5.99587	5.97047 (.00201491)	5.99514 (.00148941)	5.99392	5.99454 (.00101168	5.94985 (.00535463)	5.99634	5.95558 (.00372918
$\overline{\varepsilon}$		-0.11514*		.577955 (.00926255)	.354963	.561356 (.00915111)	1.00186 (.00848644	1.05171	.997559 (.00829356	0.897865 (.028212)	.446082	.863642 (.026377
SD_{ε}		0.5962*										
$\hat{\sigma}$	.632454		.609226	.139149		.136228	.074857		.070791	.255856		.240643
R^2	.999633		.999815	.999982		.999991	.999995		.999998	.999940		.999972
$\overline{R}^2$	.999631		.999814	.999982		.999991	.999995		.999998	.999940		.999972
a = 0; b =												
$\hat{a}_1$	5.52774 (.109689)	5.65932*	5.61763 (.073122)	5.54754 (.023621)	5.56091	5.58550 (.015333)	6.11841 (.046797)	7.05299	6.19783 (.031730)	4.92059 (.013254)	4.95514	4.90654 (.00946667
$\hat{a}_2$	6.03268 (6.012131)	5.99820*	6.02973 (.00845466)	6.03124 (.0026121)	6.01251	6.02851 (.00177200)	6.04734 (0.00479899	6.02649	6.04240 (.00315681	6.01179 (.00137324)	6.00937	6.01241 (.000908387)
$\overline{\varepsilon}$		0.12162*		.556861 (.00872498)	.747127	.561811 (.00855669	0.865588 (.026349)	2.35420	.879100 (.024946)	0.965583 (.00775864)	0.929791	.978143 (.00743182)
SD_{ε}		0.57904*										
$\hat{\sigma}$	.610329		.593214	.131419		.124324	.240402		.220053	.068572		.063064
R^2	.999669		.999835	.999985		.999993	.999949		.999977	.999996		.999998
$\overline{R}^2$	.999667		.999834	.999985		.999993	.999948		.999977	.999996		.999998

* Mean and Standard Deviation of the 98 coefficients.

Table 4.2 Double Normal Disturbances												
				Null CDF								
				$\Phi(z)$			$\Phi[z \mid z<0]$			$\Phi[z \mid z>0]$		
Sample	WLS	With errors	GLS	WLS	With errors	GLS	WLS	With errors	GLS	WLS	With errors	GLS
a = ∞; b = ∞												
$\hat{a}_1$	4.54074 (0.256056)	3.27177	4.41851 (.169791)	4.67234 (.026849)	27.2954	4.66244 (.017808))	6.38822 (.052650)	36.5727	6.41543 (.040802)	2.97985 (.091290)	.945480	2.95489 (.061955)
$\hat{a}_2$	6.05146 (0.028318)	6.34436	6.07083 (.09632	6.03474 (.00209977)	.648695	6.03596 (.00206470)	6.04962 (.00516297)	3.43542	6.05008 (.00360483)	6.02126 (.00920922)	.935066	6.02570 (.00611203)
$\overline{\varepsilon}$		-0.76089*		1.41443 (.010591)	29.6178	1.42101 (.010594)	2.30300 (.030346)	32.0685	2.33707 (.030929)	2.25755 (.054961)	38.0683	2.24986 (.051535)
SD_{ε}		1.74878*										
$\hat{\sigma}$	1.42474		1.37745	.149294		0.143715	.259757		.252195	.461855		.422669
R^2	.998180		.999078	.999980		.999990	.999940		.999969	.999810		.999913
$\overline{R}^2$	.998171		.999073	.999980		.999990	.999939		.999969	.999808		.999913
a = ∞; b =												
$\hat{a}_1$	3.96629 (.215400)	2.60627	3.77541 (.147230)	4.66244 (.017808)	29.6918	3.98285 (.011416)	5.49344 (.046823)	44.5084	5.51133 (.036442)	2.50365 (.075550)	15.3178	2.48111 (.050490)
$\hat{a}_2$	6.02401 (.024280)	6.34541	6.04578 (.017023)	6.03596 (.00206470)	.652111	6.02237 (.00131968)	6.03012 (.00462588)	2.06023	6.03090 (.00321192)	6.01113 (.00748642)	-.064950	6.01470 (.00493462)
$\overline{\varepsilon}$		-0.87848*		1.42101 (.010594)	.18.0954	1.22841 (.00675536)	1.97821 (.027278)	32.6284	2.00402 (.027304)	1.97263 (.045389)	26.9504	1.96130 (.041791)
SD_{ε}		1.63050*										
$\hat{\sigma}$	1.22156		1.19442	.143715		.092153	.232699		.224913	.376362		.343103
R^2	.998638		.999285	.999990		.999996	.999951		.999975	.999871		.999941
$\overline{R}^2$	.998631		.999281	.999990		.999996	.999950		.999975	.999870		.999941
a = 0; b =												
$\hat{a}_1$	5.34115 (.211037)	3.95530	5.19634 (.142125)	5.41397 (.016592)	32.2646	5.39974 (.011338)	6.80672 (.052241)	45.4072	6.83609 (.040826)	4.00317 (.064897)	13.6644	3.97678 (.043457)
$\hat{a}_2$	6.05348 (.023339)	6.34548	6.07199 (.016433)	6.04441 (.00183511)	.464380	6.04583 (.00131255)	6.05675 (.00515548)	2.26946	6.05743 (.00362132)	6.03119 (.00651418)	-.269615	6.03572 (.430470)
$\overline{\varepsilon}$		-0.64171*		1.16937 (.00655186)	21.8349	1.16907 (.00661448)	1.87398 (.030158)	33.2129	1.90524 (.030519)	1.90162 (.039166)	32.8915	1.87767 (.035712)
SD_{ε}		1.53932*										
$\hat{\sigma}$	1.17425		1.15301	.092293		.091507	.259370		.253560	.326927		.298132
R^2	.998780		.999367	.999992		.999996	.999941		.99970	.999906		.999958
$\overline{R}^2$	.998773		.3999364	.999992		.999996	.999940		.999969	.999905		.999957

* Mean and Standard Deviation of the 98 coefficients.

Conclusion

Estimation procedures of a linear regression model under truncated residual distribution assumptions have been proposed, with illustration for different density families. We focussed on truncation at zero residuals, an assumption usually encountered in stochastic frontier models.

1. Albeit the variety of procedures tested, the methods only give acceptable answer to a single residual environment – or with the majority of residual randomness coming from the truncated residuals.

2. All the methods perform satisfactorilly in identifying whether (single) truncation exists or not and of which type for normal, exponential and sometimes also the uniform residuals. Inverse order estimation performed as accurately as the direct method in model identification – but standard errors appear smaller for the former if one is willing to accept the standard ones.

3. For the normal, the application of replicated moments to the inverse order method formulation outperformed – exhibits smaller standard errors of the parameter estimates – the (simple) inverse order regression procedure.

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