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El Qalli Yassine

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EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 299 3523
Fax: +322 299 3523
www.eeri.eu

Term Structure Equations Under Benchmark Framework

Yassine EL QALLI*
Department of Mathematics, Faculty of Sciences Semlalia
Cadi Ayyad University, BP 2390, Marrakesh, Morocco
y.elqalli@ucam.ac.ma

Abstract

This paper makes use of an integrated benchmark modeling framework that allows us to derive term structure equations for bond and forward prices. The benchmark or numeraire is chosen to be the growth optimal portfolio (GOP). For deterministic short rate the solution of the bond term structure equation coincides with the explicit formula obtained in Platen(2005). The resulting term structure equations are used to explain moves in bond and forward prices by introducing GOP as a factor and therefore constructing a hedge portfolio for bond consisting of units of the GOP and the saving account. The paper also derives an affine term structure equation for forward price in term of the GOP factor. In the case of stochastic short rate we restrict our selves to give only a term structure equation for the bond price.

JEL Classification. E43, G13.
Key words and phrases. Term structure, Benchmark approach, GOP, Forward price, bond.

1 Introduction

A rich literature has now emerged to understand what moves interest rate. Understanding what moves bonds is important for several reasons. One of these reasons is forecasting. Yields on long-maturity bonds are expected values of average future short yields, at least after an adjustment for risk. This means that the current yields curve contains information about the future path of the economy. Monetary policy is a second reason for studying the bond prices. In most industrialized countries, the central bank seems to be able to move the short maturity of the bond yield curve. For a given state of the economy, a model of the bond yield curve helps to understand how movements

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at the short maturity translate into longer-term yields. This involves understanding both how the central bank conducts policy and how the transmission mechanism works.

Derivative pricing and hedging provide an other reason. For example, coupon bonds are priced as baskets of coupon payments weighted by the price of a zero-coupon bond that matures on the coupon date. Banks need to manage the risk of paying short-term interest rates on deposits while receiving long-term interest rates on loans. Hedging strategies involve contracts that are contingent on future short rates, such as swap contracts. To compute these strategies, banks need to know how the price of these derivative securities depends on the state of the economy.

Despite efforts managed to explain what moves interest rates, there is still no commonly accepted interest rate model. In the literature, the risk neutral pricing formula gives the price of a zero-coupon bond basing on the dynamics of the short rate, which is a quantity controlled by the monetary authority. Therefore, bonds are regarded as derivatives of the "underlying" short rate. In this paper we derve term structure equations for bond and forward proces especially in the detereministic short rate case. In our purpose the growth optimal portfolio plays a central role since the discounted GOP is used as the underlying security and the bond is viewed as a derivative on the GOP . The dynamic of the GOP is determined by the short rate and the market price of risk which is the GOP volatility. Unfortunately, volatility does not have a major economic interpretation and is difficult to observe. However, the dynamics of the discounted GOP is uniquely determined by the net market trend which measures the market activity. So, we prefer to choose directly the discounted GOP as a factor or underlying security. As a consequence, the resulting term structure equation is simpler than that of the risk neutral setting, and have an explicit solution which coincides with the explicit formula obtained in Platen (2006) for deterministic short rate. Based on the benchmarked forward probability measure introduced in Eddahbi and El Qalli (2008), the paper also derives a term structure equation for the forward price and shows that the solution of this equation can be expressed in an affine nature in term of the GOP factor.

The organization of the paper is as follows. Section 2 recalls some results on the benchmark framework. In section 3 we establish the term structure equation (for deterministic short rate) for bond price and construct a hedge portfolio for bond price. Section 4 is devoted to derive the term structure equation for the forward price. The last section is devoted to derive a bond term structure equation in the case of stochastic short rate.

2 Background on Benchmark Framework

2.1 Security Accounts

We consider a continuous financial market where the uncertainty is modeled by n independent standard Wiener processes $W^k = \{W_t^k, t \in [0,T]\}$, $k \in \{1, \cdots, n\}$, we note $W = \{W_t =$

$(W_t^1, \ldots, W_t^n), t \in [0,T]\}$ to be the vector of the n wiener processes. These are defined on a filtered probability space $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$ with finite time horizon T, fulfilling the usual conditions. The filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ models the evolution of market information over time, where $\mathcal{F}_t$ denotes the information available at time $t \in [0,T]$.

The market comprises $n+1$ primary security accounts. These include a saving account of the domestic currency $S^0 = \{S_t^0, t \in [0,\infty)\}$, which is locally riskless primary security account whose differential equation is given by,

$$dS_t^0 = S_t^0 r_t dt \tag{2.1}$$

for $t \in [0,T]$ with $S_0^0 = 1$. The domestic short rate process $r = \{r_t, t \in [0,T]\}$ characterizes then the evolution of the time value of the domestic currency.

The market also includes n nonnegative, risky primary security account processes $S^j = \{S_t^j, t \in [0,T]\}$, $j \in \{1,2,\ldots,n\}$, each of which contains units of one type of security and expressed in units of the domestic currency. This might be, for instance, a cum-dividend share price or the value of foreign savings account, expressed in units of the domestic currency.

To specify the dynamics of continuous primary securities in the given market, we assume that S_t^j is the unique solution of the stochastic differential equation

$$dS_t^j = S_t^j \left(a_t^j dt + \sum_{k=1}^{n} b_t^{j,k} dW_t^k \right) \tag{2.2}$$

for $t \in [0,\infty)$ with $S_0^j > 0$, $j \in \{1,2,\ldots,n\}$. Here the jth appreciation rate a_t^j is the expected return at time t that an investor receives for holding the jth primary security in the denomination of the domestic currency. We assume that $a^j = \{a_t^j, t \in [0,T]\}$, $j \in \{1,2,\ldots,n\}$, is a predictable process such that $\int_0^T \sum_{j=0}^n |a_s^j| ds < \infty$ almost surely, for all $T \in [0,\infty)$.
The j,kth volatility $b_t^{j,k}$ measures at time t the proportional fluctuations of the value of the jth primary security account with respect to the kth Wiener process. We suppose that $b_t^{j,k}$ is a given predictable process that satisfies the integrability condition $\int_0^T \sum_{j=1}^n \sum_{k=1}^n \left(b_t^{j,k}\right)^2 dt < \infty$ almost surely, for all $j,k \in \{1,2,\ldots,n\}$ and $T \in [0,\infty)$.

2.2 Self-Financing Strategies and Portfolios

In the given continuous financial market one is allowed to form portfolios of primary security accounts. We call a predictable stochastic process $\phi = \{\phi_t = (\phi_t^0, \ldots, \phi_t^n)^\top, t \in [0,T]\}$ a strategy if for each $j \in \{0,1,\ldots,n\}$ the Itô integral $\int_0^t \phi_s^j dS_s^j$ exists and such that the portfolio process $V^\phi = \{V_t^\phi, t \in [0,T]\}$ is characterized by the linear combination $V_t^\phi = \sum_{j=0}^n \phi_t^j S_t^j$ for all $t \in [0,T]$. Here ϕ_t^j is the number of units of the jth primary security account that are held at time $t \in [0,T]$ in the corresponding portfolio, $j \in \{0,1,\ldots,n\}$.

Definition 2.1 *A strategy ϕ and the corresponding portfolio V^{ϕ} are said to be self-financing if*

$$dV_t^{\phi} = \sum_{j=0}^{n} \phi_t^j dS_t^j \tag{2.3}$$

for all $t \in [0,T]$.

This means that changes in portfolio value are exactly matched by corresponding gains or losses from trade in the primary security accounts.
Now, let us introduce the notion of market price of risk. First, we assume that no primary security account is redundant in the sense that it cannot be expressed as a linear combination of other primary security accounts. The following assumption avoids redundant primary security accounts.

Assumption 2.2 *The volatility matrix $[b_t^{j,k}]_{j,k}^n$ is invertible for Lebesgue-almost every $t \in [0,T]$.*

Assumption 2.2 allows us to introduce the kth market price of risk θ_t^k with respect to the kth trading uncertainty, which is the kth Wiener process W^k, via the equation

$$\theta_t^k = \sum_{j=1}^{n} b_t^{-1j,k} \left(a_t^j - r_t\right),$$

for $t \in [0,T]$ and $k \in \{1, 2, \dots, n\}$. Then we can rewrite (2.2) for the jth primary security account in the form

$$dS_t^j = S_t^j \left(r_t dt + \sum_{k=1}^{n} b_t^{j,k} (\theta_t^k dt + dW_t^k) \right)$$

for $t \in [0,T]$ and $j \in \{1, 2, \dots, n\}$. Note that we have just parameterized (2.2) in terms of the market price of risk. For a given self-financing strategy ϕ the value of the corresponding portfolio V_t^{ϕ} satisfies the SDE

$$dV_t^{\phi} = V_t^{\phi} r_t dt + \sum_{k=1}^{n} \sum_{j=0}^{n} \phi_t^j S_t^j b_t^{j,k} (\theta_t^k dt + dW_t^k) \tag{2.4}$$

for $t \in [0,T]$.

On the other hand, for a given strategy ϕ, we introduce the fraction $\pi_{\phi,t}^j$ of the value of a corresponding strictly positive portfolio, which at time t is invested in the jth primary security account, that is $\pi_{\phi,t}^j = \phi_t^j \frac{S_t^j}{V_t^{\phi}}$ for $t \in [0,T]$ and $j \in \{0, 1, \dots, n\}$. Note that these proportions always add up to one, that is $\sum_{j=0}^{n} \pi_{\phi,t}^j = 1$ for all $t \in [0,T]$. Now we can parameterize (2.4) in term of the fractions to obtain

$$dV_t^{\phi} = V_t^{\phi} \left(r_t dt + \sum_{k=1}^{n} \sum_{j=0}^{n} \pi_{\phi,t}^j b_t^{j,k} (\theta_t^k dt + dW_t^k) \right)$$

for $t \in [0,T]$.

2.3 Growth Optimal Portfolio

We now introduce the growth optimal portfolio (GOP) with value $V_t^{\phi_*}$ at time $t \in [0, T]$, see Platen (2006). The GOP is the portfolio that maximizes the expected log-utility $E(\log(V_s^{\phi})|\mathcal{F}_t)$ from terminal wealth for all $t \in [0, s]$ and $s \in [0, T]$. Its strategy $\phi_* = \{\phi_{*t} = (\phi_{*t}^0, \phi_{*t}^1, \ldots, \phi_{*t}^n)^\top, t \in [0, T]\}$ follows directly from solving the first order condition for log-utility maximization problem. The resulting GOP satisfies the SDE

$$dV_t^{\phi_*} = V_t^{\phi_*} \left(r_t dt + \sum_{k=1}^{n} \theta_t^k (\theta_t^k dt + dW_t^k) \right)$$

for $t \in [0, T]$, see platen (2006). Obviously the GOP is uniquely determined up to its initial value $V_0^{\phi_*}$, and its dynamics is fully characterized by the market price for risk θ_t^k, $k \in \{1, 2, \ldots, n\}$, and the short rate r_t for $t \in [0, T]$. It can seen that the volatilities θ_t^k, $k \in \{1, 2, \ldots, n\}$, of the GOP are the corresponding market price of risk. This structure of the GOP is of crucial importance for understanding the typical dynamics of the market.

Now, the discounted GOP is defined by $\bar{V}_t^{\phi_*} = \dfrac{V_t^{\phi_*}}{S_t^0}$, the corresponding dynamics are

$$d\bar{V}_t^{\phi_*} = \bar{V}_t^{\phi_*} |\theta_t| \left(|\theta_t| dt + d\hat{W}_t \right)$$

where $|\theta_t|^2 = \theta_t \theta_t^\top$ is the risk premium of the GOP which is the square of the total market price of risk, and $d\hat{W}_t = \dfrac{1}{|\theta_t|} \sum_{k=1}^{n} \theta_t^k dW_t^k$.

Note that by discounting the GOP by the saving account we have separated the impact of the short rate from that of the market price of risk

The discounted GOP drift α_t at time $t \in [0, T]$, which we also refer to as *Net Market Trend*, is of the form

$$\alpha_t = \bar{V}_t^{\phi_*} |\theta_t|^2$$

this leads to

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{V}_t^{\phi_*}}}$$

and

$$d\bar{V}_t^{\phi_*} = \alpha_t dt + \sqrt{\alpha_t \bar{V}_t^{\phi_*}} d\hat{W}_t. \tag{2.5}$$

We emphasize that α_t is an observable financial quantity. Similar to volatility it measures market activity. We note also that the parameter process $\alpha = \{\alpha_t, t \in [0, \infty)\}$ can be freely specified as a predictable stochastic process such that the SDE (2.5) has a unique strong solution. Furthermore, it is important to realize that (2.5) describes the SDE of a time transformed squared Bessel process of dimension four. For more details see Platen and Heath (2006).

2.4 Fair Pricing

In principle, one has the freedom to choose any strictly positive numeraire or benchmark as reference unit. Throughout the following we use the GOP as numeraire. The choice of the GOP as numeraire has important advantages over alternatives, because this is the only choice, where it is not necessary to perform a measure transformation when pricing derivatives in incomplete market that has an equivalent risk neutral martingale measure, see Long (1990).
For a portfolio V^{ϕ} we introduce its benchmarked portfolio

$$\hat{V}_t^{\phi} = \frac{V_t^{\phi}}{V_t^{\phi_*}}$$

at time $t \in [0, T]$. By application of the Itô formula the benchmarked portfolio $\hat{V}_t^{\phi}$ satisfies the SDE

$$\hat{V}_t^{\phi} = \sum_{k=1}^{n} \sum_{j=0}^{n} \phi_t^j \hat{S}_t^j (b_t^{j,k} - \theta_t^k) dW_t^k$$

where $\hat{S}_t^j = \frac{S_t^j}{V_t^{\phi_*}}$ for $t \in [0, T]$.

Now, we introduce a general framework than what is provided by standard risk neutral approach, see Platen and Heath (2006). By using conditional expectations with respect to the real world probability measure P we introduce the following concept of fair pricing.

Definition 2.3 *A price process $U = \{U_t, t \in [0, T]\}$, with $E\left(\frac{|U_t|}{V_t^{\phi_*}}\right) < \infty$ for $t \in [0, T]$, is called fair if the corresponding benchmarked price process $\hat{U} = \{\hat{U}_t = \frac{U_t}{V_t^{\phi_*}}, t \in [0, T]\}$, forms an $(\mathcal{F}, P)$-martingale, that is*

$$\hat{U}_t = E(\hat{U}_s | \mathcal{F}_t) \tag{2.6}$$

for $0 < t \leq s \leq T$.

Consequently, for a fair price process its last observed benchmarked value is the best forecast of any of its future benchmarked values.

In this setting let us define a contingent claim H_T that matures at T as an $\mathcal{F}_T$-measurable payoff with $E\left(\frac{|H_T|}{V_T^{\phi_*}}\right) < \infty$ for all $t \in [0, T]$. In order to value this contingent claim, a corresponding price process $U^{H_T} = \{U_t^{H_T}, t \in [0, T]\}$ must satisfy the condition $U_T^{H_T} = H_T$ at T. By the martingale property (2.6) the contingent claim price $U_t^{H_T}$ of H_T, when expressed in units of the domestic currency, is at time $t \in [0, T]$ obtained by the fair pricing formula

$$U_t^{H_T} = V_t^{\phi_*} E\left(\frac{U_T^{H_T}}{V_T^{\phi_*}} | \mathcal{F}_t\right) \tag{2.7}$$

for $t \in [0, T]$.

2.5 Zero Coupon Bond and Actuarial Pricing

If the maturity date T is fixed and the payoff equals one unit of the domestic currency, then we obtain by the fair pricing formula (2.7) the price $p(t,T)$ of a corresponding zero-coupon bond at time $t \in [0,T]$. This price is given by the equation

$$p(t,T) = V_t^{\phi_*} E\left(\frac{1}{V_T^{\phi_*}}|\mathcal{F}_t\right)$$

the corresponding benchmarked zero-coupon bond price

$$\hat{p}(t,T) = \frac{p(t,T)}{V_t^{\phi_*}} = E\left(\frac{1}{V_T^{\phi_*}}|\mathcal{F}_t\right) \tag{2.8}$$

for $t \in [0,T]$. The benchmarked zero-coupon bond price process $\hat{p}(\cdot,T) = \{\hat{p}(t,T), t \in [0,T]\}$ is then an $(\mathcal{F},P)$-martingale. So, it is reasonable to assume that there exists for each $t \in [0,T]$ and $k \in \{1,2,\dots,n\}$ a unique predictable kth benchmarked bond volatility $\sigma_k(t,T)$ such that

$$d\hat{p}(t,T) = \hat{p}(t,T)\sum_{k=1}^{n}\sigma_k(t,T)dW_t^k \tag{2.9}$$

which is equivalent to

$$d\hat{p}(t,T) = \hat{p}(t,T)\sigma(t,T)dW_t \tag{2.10}$$

where $\sigma(t,T) = (\sigma_1(t,T),\dots,\sigma_n(t,T))$ and thus

$$\hat{p}(t,T) = \hat{p}(0,T)\exp\left\{-\sum_{k=1}^{n}\left(\int_0^t \frac{(\sigma_k(s,T))^2}{2}ds - \int_0^t \sigma_k(s,T)dW_t^k\right)\right\} \tag{2.11}$$

for each $t \in [0,T]$, then we obtain via the Itô formula

$$dp(t,T) = p(t,T)\left(r_t dt + \sum_{k=1}^{n}(\theta_t^k - \sigma_k(t,T))[\theta_t^k dt + dW_t^k)]\right). \tag{2.12}$$

We assume now that the short rate r_t is deterministic. The fair pricing formula leads to

$$p(t,T) = V_t^{\phi_*} E\left(\frac{1}{V_T^{\phi_*}}|\mathcal{F}_t\right) = \exp\left\{-\int_t^T r_s ds\right\} E\left(\frac{\bar{V}_t^{\phi_*}}{\bar{V}_T^{\phi_*}}|\mathcal{F}_t\right). \tag{2.13}$$

By using the first negative moment of a squared Bessel process of dimension of dimension four (see Platen & Heath (2006)) we have

$$E\left((\bar{V}_T^{\phi_*})^{-1}|\mathcal{F}_t\right) = (\bar{V}_t^{\phi_*})^{-1}\left(1 - \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2}ds}\right\}\right). \tag{2.14}$$

Therefore, we obtain the Platen's explicit formula (see Platen & Heath (2006)) for the zero-coupon bond

$$p(t,T) = \exp\left(-\int_t^T r_s ds\right)\left(1 - \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2} ds}\right\}\right) \tag{2.15}$$

Note that we have supposed that r_t is deterministic. But we should prove later that this result remains true for stochastic short rates.

Now, let us consider at the fixed maturity date T a random $\mathcal{F}_T$-measurable payoff $H > 0$, which is independent of the GOP value $V_T^{\phi_*}$. For instance, this could be life insurance claim or a payoff based on a weather index. Such a claim may be independent of the GOP. To be precise, we assume that the expectation of the benchmarked payoff $E\left(\left|\frac{H}{V_T^{\phi_*}}\right|\right) < \infty$ is finite. According to the fair pricing formula and using the fact that H is independent of $V_T^{\phi_*}$ we have

$$U_t^H = V_t^{\phi_*} E\left(\left.\frac{H}{V_T^{\phi_*}}\right| \mathcal{F}_t\right) = V_t^{\phi_*} E\left(\left.\frac{1}{V_T^{\phi_*}}\right| \mathcal{F}_t\right) E\left(H \middle| \mathcal{F}_t\right).$$

By using now the fair zero-coupon bond price $p(t,T)$, it follows the widely used actuarial pricing formula

$$U_t^H = p(t,T) E\left(H \middle| \mathcal{F}_t\right).$$

Under this formula one computes the conditional expectation of a future cash flow at time T and discounts it back to the present time t by using the corresponding fair zero-coupon bond price. This takes into account the evolution of the time value of money.

3 Bond Price Term Structure Equation in Term of GOP: Deterministic Short rate Case

In this section we view the bond price as derivative of the GOP. In contrast to the risk neutral setting where a natural starting point is to give an a prior specification of the dynamics of the short interest rate, the advantage here is the fact that the dynamics of the GOP are uniquely determined by (2.5). Second, in the risk neutral setting the number of exogenously given traded assets excluding the risk free asset equals zero, whereas the number of random source equals more than one. So, the market in the risk neutral setting is expected to be arbitrage free but note complete. Here, we have to explain the bond price by the GOP which is a risky portfolio. So, in the benchmark framework we have to work under a *"Complete" Zero-Coupon Bond Market*. To make this idea more correct we set the following assumption

Assumption 3.1 *We assume that for every T, the price of a zero-coupon bond price has the form*

$$p(t,T) = H_p(t, \bar{V}_t^{\phi_*}, T) \tag{3.16}$$

and the net market trend has the form

$$\alpha_t = \bar{\alpha}(t, \bar{V}_t^{\phi_*}) \tag{3.17}$$

where H_p is a smooth function of the three real variables, and $\bar{\alpha}$ is sufficiently smooth function.

At the time of maturity a zero-coupon bond is of course worth exactly \$1, so we have the relation

$$H_p(T, v, T) = 1 \quad \text{for all } v.$$

Assumption 3.1 implies we want to explain moves in the zero-coupon bond only by the source of randomness described by the discounted GOP $\bar{V}_t^{\phi_*}$. Consequently, Assumption 3.1 forces the short rate to be deterministic. Now, from Assumption 3.1 and the Itô formula we get the following price dynamics for the zero-coupon bond.

Proposition 3.1 *Suppose the zero-coupon bond price satisfies Assumption (3.1). Then H_p satisfies the term structure equation*

$$\begin{cases} \dfrac{\partial H_p}{\partial t} + \dfrac{1}{2}\alpha_t v \dfrac{\partial^2 H_p}{\partial v^2} = r_t H_p \\ H_p(T, v, T) = 1 \end{cases} \tag{3.18}$$

Proof. Applying Itô formula to Equation (3.16) and using the GOP dynamics (2.5) we get the dynamics of the zero-coupon bond price under the real world probability P as follows

$$dp(t,T) = \left\{ \frac{\partial H_p}{\partial t} + \alpha_t \frac{\partial H_p}{\partial v} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*} \frac{\partial^2 H_p}{\partial v^2} \right\} dt + \frac{\partial H_p}{\partial v} \sqrt{\alpha_t \bar{V}_t^{\phi_*}}\, d\hat{W}_t. \tag{3.19}$$

Note that the zero-coupon is not martingale under the real world probability. However, the martingale property holds for the benchmarked zero-coupon bond. So, applying the Itô formula to benchmarked zero-coupon (2.8) we get

$$\begin{aligned} d\hat{p}(t,T) &= d\left(\frac{p(t,T)}{V_t^{\phi_*}}\right) = d\left(\frac{H_p(t, \bar{V}_t^{\phi_*}, T)}{V_t^{\phi_*}}\right) \\ &= H_p(t, \bar{V}_t^{\phi_*}, T)\, d\left(\frac{1}{V_t^{\phi_*}}\right) + \frac{1}{V_t^{\phi_*}} dH_p(t, \bar{V}_t^{\phi_*}, T) \\ &\quad + \left\langle H_p(\cdot, \bar{V}_\cdot^{\phi_*}, T), \frac{1}{V_\cdot^{\phi_*}} \right\rangle_t . \end{aligned}$$

On the other hand we have

$$d\left(\frac{1}{V_t^{\phi_*}}\right) = -\frac{1}{V_t^{\phi_*}}\left[r_t dt + |\theta_t| d\hat{W}_t\right]$$

and therefore

$$d\left\langle H_p(\cdot, \bar{V}_\cdot^{\phi_*}, T), \frac{1}{V_\cdot^{\phi_*}}\right\rangle_t = -\frac{\partial H_p}{\partial v}\frac{|\theta_t|}{V_t^{\phi_*}}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}dt = -\frac{\partial H_p}{\partial v}\frac{\alpha_t}{V_t^{\phi_*}}dt$$

where $\langle\cdot,\cdot\rangle_t$ denotes the quadratic variation. Hence, the benchmarked zero-coupon bond becomes

$$\begin{aligned} d\hat{p}(t,T) &= -\frac{H_p}{V_t^{\phi_*}}\left[r_t dt + |\theta_t| d\hat{W}_t\right] + \frac{1}{V_t^{\phi_*}}\left[\frac{\partial H_p}{\partial t} + \alpha_t \frac{\partial H_p}{\partial v} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*}\frac{\partial^2 H_p}{\partial v^2}\right] dt \\ &\quad + \left\{\frac{1}{V_t^{\phi_*}}\frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}\right\} d\hat{W}_t - \frac{\partial H_p}{\partial v}\frac{\alpha_t}{V_t^{\phi_*}}dt \\ &= \frac{1}{V_t^{\phi_*}}\left\{-H_p r_t + \frac{\partial H_p}{\partial t} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*}\frac{\partial^2 H_p}{\partial v^2}\right\} dt \\ &\quad + \frac{1}{V_t^{\phi_*}}\left\{\frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}} - H_p|\theta_t|\right\} d\hat{W}_t \end{aligned}$$

and the result follows from the fact that the benchmarked zero-coupon bond with maturity T is martingale under the real world probability measure P , so its drift term must be zero. ∎

Now, taking partial derivatives for (2.15) with respect to $(t, v) \in (0, T) \times (0, \infty)$ we get

$$\frac{\partial p(t,T)}{\partial t} = r_t p(t,T) + \left(\exp\left\{-\int_t^T r_s ds\right\} - p(t,T)\right)\frac{\bar{V}_t^{\phi_*}\frac{\alpha_t}{2}}{\left(\int_t^T \frac{\alpha_s}{2}ds\right)^2} \tag{3.20}$$

$$\frac{\partial^2 p(t,T)}{\partial v^2} = \left(\exp\left\{-\int_t^T r_s ds\right\} - p(t,T)\right)\frac{-1}{\left(\int_t^T \frac{\alpha_s}{2}ds\right)^2}. \tag{3.21}$$

Multiplying equation (3.21) by $\frac{1}{2}\alpha_t v$ and adding it to (3.20) we get

$$p(t,T) = H_p(t, \bar{V}_t^{\phi_*}, T) = \exp\left\{-\int_t^T r_s ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2}ds}\right\}\right)$$

Proposition 3.2 *The explicit Platen's formula (2.15) given by*

$$p(t,T) = H_p(t, \bar{V}_t^{\phi_*}, T) = \exp\left\{-\int_t^T r_s ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2}ds}\right\}\right) \tag{3.22}$$

solves the term structure equation (5.44).

3.1 Hedging Zero-Coupon Bond by the GOP

Our aim is to construct a self-financing hedge portfolio consisting of ϕ_t^0 units of the saving account S_t^0 and ϕ_t^1 units of the GOP $V_t^{\phi_*}$ such that the value of that portfolio V_t^{ϕ} equals to the price of the zero-coupon bond i.e.

$$V_t^{\phi} = p(t,T) = H_p(t, \bar{V}_t^{\phi_*}, T).$$

According to (5.46) and the term structure equation (5.44) we have

$$dp(t,T) = \left\{ r_t H_p + \alpha_t \frac{\partial H_p}{\partial v} \right\} dt + \frac{\partial H_p}{\partial v} \sqrt{\alpha_t \bar{V}_t^{\phi_*}} d\hat{W}_t. \tag{3.23}$$

On the other hand

$$V_t^{\phi} = \phi_t^0 S_t^0 + \phi_t^1 V_t^{\phi_*} = \phi_0^0 S_0^0 + \phi_0^1 V_0^{\phi_*} + \int_0^t \phi_s^0 dS_s^0 + \int_0^t \phi_s^1 dV_s^{\phi_*}$$

and

$$dV_t^{\phi_*} = d(\bar{V}_t^{\phi_*} S_t^0) = \bar{V}_t^{\phi_*} S_t^0 r_t dt + S_t^0 d\bar{V}_t^{\phi_*}$$

which give an other expression for $p(t,T)$

$$\begin{aligned} dp(t,T) = dV_t^{\phi} &= \phi_t^0 r_t S_t^0 dt + \phi_t^1 dV_t^{\phi_*} \\ &= \phi_t^0 r_t S_t^0 dt + \phi_t^1 \left[\bar{V}_t^{\phi_*} S_t^0 r_t dt + S_t^0 d\bar{V}_t^{\phi_*} \right] \\ &= \left\{ \left[\phi_t^0 + \phi_t^1 \bar{V}_t^{\phi_*} \right] r_t S_t^0 + \alpha_t S_t^0 \phi_t^1 \right\} dt + \phi_t^1 S_t^0 \sqrt{\alpha_t \bar{V}_t^{\phi_*}} d\hat{W}_t. \end{aligned} \tag{3.24}$$

Comparing Equations (3.23) and (3.24) we get

$$\begin{aligned} \phi_t^1 S_t^0 \sqrt{\alpha_t \bar{V}_t^{\phi_*}} &= \frac{\partial H_p}{\partial v} \sqrt{\alpha_t \bar{V}_t^{\phi_*}} \\ \left[\phi_t^0 + \phi_t^1 \bar{V}_t^{\phi_*} \right] r_t S_t^0 + \alpha_t S_t^0 \phi_t^1 &= r_t H_p + \alpha_t \frac{\partial H_p}{\partial v}. \end{aligned}$$

It follows that

$$\begin{aligned} \phi_t^1 &= \frac{1}{S_t^0} \frac{\partial H_p}{\partial v} \\ \phi_t^0 &= \frac{1}{S_t^0} \left[H_p - \frac{\partial H_p}{\partial v} \bar{V}_t^{\phi_*} \right] \end{aligned}$$

substituting the expression of H_p from (3.22), we have the following explicit formula for ϕ_t^1

$$\phi_t^1 = \frac{1}{S_t^0} \frac{\partial H_p}{\partial v} = \frac{\exp\left\{ -\int_0^T r_s ds \right\}}{\int_t^T \frac{\alpha_s}{2} ds} \exp\left\{ -\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2} ds} \right\}.$$

Furthermore from (3.22) we have

$$\frac{\partial H_p}{\partial v} = \frac{\exp\left\{-\int_t^T r_s ds\right\} - H_p}{\int_t^T \frac{\alpha_s}{2} ds}$$

so that we can rewrite ϕ_t^0 as

$$\begin{aligned}\phi_t^0 &= \exp\left\{-\int_0^t r_s ds\right\}\left[H_p - \bar{V}_t^{\phi_*}\left\{\frac{\int_t^T r_s ds - H_p}{\int_t^T \frac{\alpha_s}{2} ds}\right\}\right] \\ &= \exp\left\{-\int_0^t r_s ds\right\}\left[p(t,T) - \bar{V}_t^{\phi_*}\left\{\frac{\int_t^T r_s ds - p(t,T)}{\int_t^T \frac{\alpha_s}{2} ds}\right\}\right].\end{aligned}$$

Now, with the quantity

$$A_t = A_0 + \int_0^t \alpha_s ds$$

we can rewrite ϕ_t^1, ϕ_t^2 and the zero-coupon bond to get the following explicit formulae

$$\begin{aligned}\phi_t^1 &= \frac{\exp\left\{-\int_0^T r_s ds\right\}}{\frac{A_T - A_t}{2}} \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\frac{A_T - A_t}{2}}\right\} \\ \phi_t^0 &= \exp\left\{-\int_0^t r_s ds\right\}\left[p(t,T) - \bar{V}_t^{\phi_*} \exp\left\{\frac{\int_t^T r_s ds - p(t,T)}{\frac{A_T - A_t}{2}}\right\}\right] \\ p(t,T) &= \exp\left\{-\int_t^T r_s ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\frac{A_T - A_t}{2}}\right\}\right).\end{aligned}$$

Here, A_t can be interpreted as the underlying value at time t of the discounted GOP, where A_0 needs to be appropriately chosen as the initial underlying value at time $t = 0$. One can say that the underlying value A_t corresponds to the discounted wealth that underlies the discounted index $\bar{V}_t^{\phi_*}$.

The advantage here i.e. in the benchmark framework is the fact that we try to understand what moves bond term structure with respect to its underlying economic conjecture. In fact, we have written the bond price as a function of the discounted GOP $\bar{V}_t^{\phi_*}$ which is learned to be a discounted market index, A_t which corresponds to the discounted wealth that underlies the discounted index, and of course the short rate which in an other economic source of randomness controlled by the central bank or monetary authority.

4 Forward Price Term Structure Equation in Term of GOP: Case of Deterministic Short Rate

A forward contract is an agreement, established at the date $t < T$, to pay or receive on settlement date T a preassigned payoff, say H, at an agreed forward price. Let us emphasize that there is no cash flow at the contract's initiation and the contract is not marked to market. We may assume, without loss of generality, that a forward contract is settled by cash on date T. Therefore, a forward contract written at time t with the underlying contingent claim H and prescribed settlement date T may be summarized by the following two basic rules : (a) a cash amount H will be received at time T, and a preassigned amount $F_H(t,T)$ of cash will be paid at time T; (b) the amount $F_H(t,T)$ should be predetermined at time t (according to the information available at this time) in such a way that the price of the forward contract at time t is zero. In fact, since nothing is paid up front, it is natural to admit that the forward contract is worthless at its initiation. We adopt the following formal definition of a forward contract.

Definition 4.1 *Let us fix $0 \le t \le T$. A forward contract written at time t on a time T contingent claim H is represented by the time T contingent claim $G_T = H - F_H(t,T)$ that satisfies the following conditions :*

1. *$F_H(t,T)$ is a $\mathcal{F}_t$-measurable random variable;*
2. *the fair price at time t of a contingent claim G_T equals zero.*

The random variable $F_H(t,T)$ is referred to as the forward price of a contingent claim H at time t for the settlement date T. The contingent claim H may be defined in particular as a preassigned amount of the underlying financial asset to be delivered at the settlement date. For instance, if the underlying asset of forward contract is a given portfolio V^ϕ issued from a given strategy ϕ, then clearly $H = V_T^\phi$. The following result expresses the forward price of a contingent claim H in terms of its fair price U_t^H and the price $p(t,T)$ of a zero-coupon bond which matures at time T.

Proposition 4.1 *The forward price $F_H(t,T)$ at time $t \le T$, for the settlement date T, of a contingent claim H equals*

$$F_H(t,T) = \frac{U_t^H}{p(t,T)} \tag{4.25}$$

Proof. By the fair pricing formula (2.7) we have

$$0 = V_t^{\phi_*} E\left(\frac{G_T}{V_T^{\phi_*}} \middle| \mathcal{F}_t \right) = V_t^{\phi_*} E\left(\frac{H - F_H(t,T)}{V_T^{\phi_*}} \middle| \mathcal{F}_t \right)$$

by using the fair zero-coupon bond price and the fair pricing formula it follows the forward price

$$F_H(t,T) = \frac{V_t^{\phi_*} E\left(\frac{H}{V_T^{\phi_*}} \middle| \mathcal{F}_t \right)}{V_t^{\phi_*} E\left(\frac{1}{V_T^{\phi_*}} \middle| \mathcal{F}_t \right)} = \frac{U_t^H}{p(t,T)}.$$

In the case when the underlying asset of the forward contract is a given fair portfolio V^{ϕ} issuing from a given strategy ϕ we have the following forward price

$$F_{V_T^{\phi}}(t,T) = \frac{V_t^{\phi}}{p(t,T)}$$

for $t \in [0,T]$. ■

Note that, if H is independent to $V_T^{\phi_*}$ then from the actuarial pricing formula we have $F_H(t,T) = E(H|\mathcal{F}_t)$. Hence, the forward price is an $(\mathcal{F}, P)$-martingale, that is, the forward price is the best forecast of the future value H. The question which arises now is; what happens if H is not necessary independent of $V_T^{\phi_*}$? to answer this question we need to introduce the notion of Benchmarked forward probability measure.

4.1 Benchmarked Forward Probability Measure

According to our knowledge, within the framework of arbitrage valuation of interest rate derivatives, the method of a forward risk adjustment was pioneered under the name of a forward risk-adjusted process by Jamshidian (1987). The formal definition of a forward probability measure was explicitly introduced in Geman (1989) under the name of forward neutral probability. In particular, Geman observed that the forward price of any financial asset follows a (local) martingale under the forward neutral probability associated with the settlement date of a forward contract. For further developments of the forward measure approach, we refer the reader in particular, to El Karoui *et al.* (1995).

Definition 4.2 *A probability measure P_T on $(\Omega, \mathcal{F}_T)$ equivalent to P, with the Radon-Nikodym derivative given by the formula:*

$$\frac{dP_T}{dP} = \frac{V_0^{\phi_*}}{V_T^{\phi_*} p(0,T)}$$

is called the benchmarked forward martingale measure for the settlement date T.

Notice that the above Radon-Nikodym derivative, when restricted to the σ-field $\mathcal{F}_t$ satisfies for every $t \in [0,T]$

$$\eta_t \stackrel{def}{=} \frac{dP_T}{dP}\Big|_{\mathcal{F}_t} = E\left(\frac{V_0^{\phi_*}}{V_T^{\phi_*} p(0,T)} \middle| \mathcal{F}_t \right) = \frac{V_0^{\phi_*} p(t,T)}{V_t^{\phi_*} p(0,T)}.$$

Now, by using equations (2.8) and (2.11), we obtain

$$\eta_t = \frac{\hat{p}(t,T)}{\hat{p}(0,T)} = \exp\left\{ -\sum_{k=1}^{n} \left(\int_0^t \frac{(\sigma_k(s,T))^2}{2} ds - \int_0^t \sigma_k(s,T) dW_t^k \right) \right\}.$$

Furthermore, the processes

$$W_t^{k,T} = W_t^k - \int_0^t \sigma_k(s,T)ds, \quad k = 1, \dots, n \tag{4.26}$$

or in vectoriel form

$$W_t^T = W_t - \int_0^t \sigma(s,T)ds \tag{4.27}$$

for all $t \in [0,T]$, follow a standard Brownian motion under the benchmarked forward measure P_T.

The next result shows that the forward price of contingent claim H which settles at time T can be easily expressed in terms of the conditional expectation under the benchmarked forward measure P_T. Denote E_{P_T} the expectation under P_T.

Proposition 4.2 *The forward price at t for the date T of a contingent claim H which settles at time T equals*

$$F_H(t,T) = E_{P_T}(H|\mathcal{F}_t) \tag{4.28}$$

provided that H is P_T integrable. In particular the forward price process $F_H(t,T)$, $t \in [0,T]$, follows a martingale under the benchmarked forward measure P_T.

Proof. The Bayes rule yields

$$\begin{aligned} E_{P_T}[H|\mathcal{F}_t] \quad &= \frac{E[\eta_T H|\mathcal{F}_t]}{E[\eta_T|\mathcal{F}_t]} = E[\eta_T \eta_t^{-1} H|\mathcal{F}_t] = \frac{V_t^{\phi_*}}{p(t,T)} E[\frac{H}{V_T^{\phi_*}}|\mathcal{F}_t] \\ &= \frac{V_t^{\phi_*}}{p(t,T)} \frac{U_t^H}{V_t^{\phi_*}} = \frac{U_t^H}{p(t,T)} \end{aligned}$$

■

If the underlying asset of a forward contract is a given fair portfolio V^ϕ, then clearly $F_H(t,T) = E_{P_T}[V_T^\phi|\mathcal{F}_t]$. The next proposition establishes a version of the actuarial pricing formula that is tailored to any contingent claim although it is not independent of the GOP V^{ϕ_*}.

Proposition 4.3 *The fair price of any contingent claim H which settles at time T is given by the following version of the actuarial pricing formula*

$$U_t^H = p(t,T) E_{P_T}[H|\mathcal{F}_t] \tag{4.29}$$

Proof. Equality (4.29) is an immediate consequence of (4.25) combined with (4.28). For a more direct proof, note that the fair price U_t^H can be reexpressed as follows

$$U_t^H = V_t^{\phi_*} E[\frac{H}{V_T^{\phi_*}}|\mathcal{F}_t] = V_t^{\phi_*} \frac{p(0,T)}{V_0^{\phi_*}} E[\eta_T H|\mathcal{F}_t].$$

An application of the Bayes rule yields

$$\begin{aligned} U_t^H &= V_t^{\phi*} \frac{p(0,T)}{V_0^{\phi*}} E_{P_T}[H|\mathcal{F}_t] E[\eta_T|\mathcal{F}_t] = V_t^{\phi*} \frac{p(0,T)}{V_0^{\phi*}} E_{P_T}[H|\mathcal{F}_t] E[\frac{V_0^{\phi*}}{V_T^{\phi*} P(0,T)}|\mathcal{F}_t] \\ &= V_t^{\phi*} E[\frac{1}{V_T^{\phi*}}|\mathcal{F}_t] E_{P_T}[H|\mathcal{F}_t] = p(t,T) E_{P_T}[H|\mathcal{F}_t] \end{aligned}$$

■

4.2 The forward price term structure equation

Modeling the entire term structure of forward prices also results in an infinite dimensional state variable. Therefore it is sometimes more convenient to model a given finite dimensional state process, and to assume that forward prices are given as functions of this state process. Similarly to the bond price, we choose the GOP as a factor and we explain moves on the forward price by the GOP dynamics.

Assumption 4.3 *We assume that for every T, the forward price can be written in the following form*

$$f(t,T) = H_f(t, \bar{V}_t^{\phi*}, T) \tag{4.30}$$

and the net market trend has the form

$$\alpha_t = \bar{\alpha}(t, \bar{V}_t^{\phi*}) \tag{4.31}$$

where H_f is a smooth function of the three real variables with the boundary condition

$$H_f(T, v, T) = h_f(T, v)$$

for all v and for an a priori given function h_f and $\bar{\alpha}$ is sufficiently smooth function.

Similarly the short rate in this case is deterministic.

Proposition 4.4 *Suppose the forward price satisfies Assumption (4.3). Then H_f satisfies the following term structure equation*

$$\begin{cases} \dfrac{\partial H_f}{\partial t} + \alpha_t \dfrac{\partial H_f}{\partial v} + \dfrac{1}{2}\alpha_t v \dfrac{\partial^2 H_f}{\partial v^2} + \dfrac{\partial H_f}{\partial v}\dfrac{\partial H_p}{\partial v}\alpha_t v = 0 \\ H_f(T, v, T) = h_f(T, v). \end{cases} \tag{4.32}$$

Proof. Applying Itô formula to Equation (4.30) and using the GOP dynamics (2.5) we get the dynamics of the forward price under the real world probability P as follows

$$dH_f(t,\bar{V}_t^{\phi_*},T) = \left\{\frac{\partial H_f}{\partial t} + \alpha_t \frac{\partial H_f}{\partial v} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*}\frac{\partial^2 H_f}{\partial v^2}\right\} dt + \frac{\partial H_f}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}} d\hat{W}_t. \tag{4.33}$$

Using the dynamics under P and noting that in this case the bond prices volatility is given by

$$\sigma_k(t,T) = \frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}\frac{\theta_t^k}{|\theta_t|}$$

we can change to the P_T measure using

$$dW_t^{k,T} = dW_t^k - \sigma_k(t,T)dt$$

and therefore

$$\begin{aligned} d\hat{W}_t &= \frac{1}{|\theta_t|}\sum_{k=1}^n \theta_t^k dW_t^k \\ &= \frac{1}{|\theta_t|}\sum_{k=1}^n \theta_t^k \left(dW_t^{k,T} + \frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}\frac{\theta_t^k}{|\theta_t|}dt\right) \\ &= \frac{1}{|\theta_t|}\sum_{k=1}^n \theta_t^k dW_t^{k,T} + \frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}dt \\ &= d\hat{W}_t^T + \frac{\partial H_p}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}}dt \end{aligned}$$

where

$$d\hat{W}_t^T = \frac{1}{|\theta_t|}\sum_{k=1}^n \theta_t^k dW_t^{k,T}.$$

The dynamics under the benchmarked forward measure P_T of $f(t,T)$ become

$$\begin{aligned} dH_f(t,\bar{V}_t^{\phi_*},T) &= \left\{\frac{\partial H_f}{\partial t} + \alpha_t \frac{\partial H_f}{\partial v} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*}\frac{\partial^2 H_f}{\partial v^2} + \frac{\partial H_f}{\partial v}\frac{\partial H_p}{\partial v}\alpha_t \bar{V}_t^{\phi_*}\right\} dt \\ &\quad + \frac{\partial H_f}{\partial v}\sqrt{\alpha_t \bar{V}_t^{\phi_*}} d\hat{W}_t^T \end{aligned}$$

and the result follows from the fact that the forward prices with maturity T are martingales under the benchmarked forward measure P_T , so its drift term must be zero. ∎

Definition 4.4 *The term structure of the forward price is said to be affine if the function H_f from (4.30) is of the following form*

$$\ln H_f(t,v,T) = A_f(t,T) + B_f(t,T)v \tag{4.34}$$

where A_f and B_f are deterministic functions.

Proposition 4.5 *Suppose that Assumption 4.3 is in force and that the function h is of the following form*

$$\ln h(T, v) = c(T) + d(T)v$$

Then the term structure of the forward price is affine, that is H_f from (4.30) can be written on the form (4.34) where A_f and B_f are given by

$$B_f(t,T) = d(T)e^{\int_t^T \alpha_s \frac{\partial H_p(s,v,T)}{\partial v} ds} \left(1 - d(T)\int_t^T \frac{1}{2}\alpha_u e^{\int_u^T \alpha_s \frac{\partial H_p(s,v,T)}{\partial v} ds} du\right)^{-1} \tag{4.35}$$

and

$$A_t(t,T) = c(T) + \int_t^T \alpha_s B_f(s,T) ds. \tag{4.36}$$

Proof. We need to show that $H_f(t, v, T)$ from (4.34) where A_f and B_f are given by (4.35) and (4.36), solves the PDE (14) that uniquely characterizes the bond prices in this setting.
Taking partial derivatives

$$\begin{aligned} \frac{\partial H_f}{\partial t} &= \left[\frac{\partial A_f}{\partial t} + \frac{\partial B_f}{\partial t} v\right] H_f \\ \frac{\partial^2 H_f}{\partial v^2} &= B_f^2 H_f \end{aligned}$$

so the equation (4.32) reduces in this case to

$$\begin{cases} \dfrac{\partial A_f(t,T)}{\partial t} + \alpha_t B_f(t,T) = 0 \\ \\ A_f(T,T) = c(T). \end{cases} \tag{4.37}$$

and

$$\begin{cases} \dfrac{\partial B_f(t,T)}{\partial t} + \dfrac{1}{2}\alpha_t B_f^2(t,T) + \alpha_t \dfrac{\partial H_p}{\partial v} B_f(t,T) = 0 \\ \\ B_f(T,T) = d(T). \end{cases} \tag{4.38}$$

Equation (4.38) is a Bernoulli differential equation with solution given by

$$B_f(t,T) = d(T)e^{\int_t^T \alpha_s \frac{\partial H_p(s,v,T)}{\partial v} ds} \left(1 - d(T)\int_t^T \frac{1}{2}\alpha_u e^{\int_u^T \alpha_s \frac{\partial H_p(s,v,T)}{\partial v} ds} du\right)^{-1}$$

and

$$A_f(t,T) = c(T) + \int_t^T \alpha_s B_f(s,T) ds$$

■

Remark 4.5 *Since we have*

$$\frac{\partial H_p(t,\bar{V}_t^{\phi_*},T)}{\partial v} = \exp\left\{-\int_t^T r_s ds\right\} \frac{1}{\int_t^T \frac{\alpha_s}{2} ds} \exp\left\{-\frac{\bar{V}_t^{\phi_*}}{\int_t^T \frac{\alpha_s}{2} ds}\right\}$$

the above proposition gives an explicit formula to the forward price in term of the GOP $\bar{V}_t^{\phi_*}$, the net market trend α_t and the short interest rate r_t.

4.3 Hedging Forward price by the GOP

In this subsection we construct a self-financing hedge portfolio consisting of π_t^0 units of the saving account S_t^0 and π_t^1 units of the GOP $V_t^{\phi_*}$ such that the value of that portfolio V_t^π equals to the forward price i.e.

$$V_t^\pi = p(t,T) = H_f(t,\bar{V}_t^{\phi_*},T).$$

According to (4.33) and the term structure equation (4.32) we have

$$dH_f(t,\bar{V}_t^{\phi_*},T) = \left\{-\frac{\partial H_f}{\partial v}\frac{\partial H_p}{\partial v}\alpha_t\bar{V}_t^{\phi_*}\right\} dt + \frac{\partial H_p}{\partial v}\sqrt{\alpha_t\bar{V}_t^{\phi_*}}d\hat{W}_t. \tag{4.39}$$

On the other hand

$$V_t^\pi = \pi_t^0 S_t^0 + \pi_t^1 V_t^{\phi_*} = \pi_0^0 S_0^0 + \pi_0^1 V_0^{\phi_*} + \int_0^t \pi_s^0 dS_s^0 + \int_0^t \pi_s^1 dV_s^{\phi_*}$$

and

$$dV_t^{\phi_*} = d(\bar{V}_t^{\phi_*} S_t^0) = \bar{V}_t^{\phi_*} S_t^0 r_t dt + S_t^0 d\bar{V}_t^{\phi_*}$$

which give an other expression for $H_f(t,\bar{V}_t^{\phi_*},T)$

$$\begin{aligned} dH_f(t,\bar{V}_t^{\phi_*},T) &= dV_t^\pi \\ &= \left\{\left[\pi_t^0 + \pi_t^1\bar{V}_t^{\phi_*}\right] r_t S_t^0 + \alpha_t S_t^0 \pi_t^1\right\} dt + \pi_t^1 S_t^0\sqrt{\alpha_t\bar{V}_t^{\phi_*}}d\hat{W}_t. \end{aligned} \tag{4.40}$$

Comparing Equations (4.39) and (4.40) we get

$$\begin{aligned} \pi_t^1 S_t^0\sqrt{\alpha_t\bar{V}_t^{\phi_*}} &= \frac{\partial H_f}{\partial v}\sqrt{\alpha_t\bar{V}_t^{\phi_*}} \\ \left[\pi_t^0 + \pi_t^1\bar{V}_t^{\phi_*}\right] r_t S_t^0 + \alpha_t S_t^0 \pi_t^1 &= -\frac{\partial H_f}{\partial v}\frac{\partial H_p}{\partial v}\alpha_t\bar{V}_t^{\phi_*}. \end{aligned}$$

It follows that

$$
\begin{aligned}
\pi_t^1 &= \frac{1}{S_t^0}\frac{\partial H_f}{\partial v} \\
\pi_t^0 &= \frac{1}{S_t^0}\left[\bar{V}_t^{\phi_*} + \frac{\alpha_t}{r_t}\left(1 + \frac{\partial H_p}{\partial v}\bar{V}_t^{\phi_*}\right)\right]\frac{\partial H_f}{\partial v}.
\end{aligned}
$$

Note that in the case of affine term structure for forward price we can get an explicit formulae for π^0 and π^1 since $\frac{\partial H_f}{\partial v}$ and $\frac{\partial H_p}{\partial v}$ can be expressed explicitly by

$$\frac{\partial H_f}{\partial v} = B_f(t,T)H_f$$

and

$$\frac{\partial H_p}{\partial v} = \frac{\exp\left\{-\int_t^T r_s ds\right\} - H_p}{\int_t^T \frac{\alpha_s}{2}ds}.$$

5 Bond Price Term Structure Equation in Term of GOP and Stochastic Short rate

In this section we view the bond price as "derivative" of the GOP and the short rate. Although the GOP dynamics are uniquely determined, we are forced to give an a prior specification of the dynamics of the short interest rate. According to Platen and Heath (2006) (see formula (10.4.17)) we choose the following dynamics for the short interest rate

$$dr_t = \left[\mu_t + \sum_{k=1}^{d}\beta_t^k\theta_t^k\right]dt + \sum_{k=1}^{d}\beta_t^k dW_t^k \tag{5.41}$$

where μ_t and $\beta_t^k, k = 1,\ldots,d$, are sufficiently regular processes.

Assumption 5.1 *We assume that for every T, the price of a zero-coupon bond price has the form*

$$p(t,T) = \tilde{H}_p(t, r_t, \bar{V}_t^{\phi_*}, T) \tag{5.42}$$

and the net market trend is of the form

$$\alpha_t = \bar{\alpha}(t, \bar{V}_t^{\phi_*}) \tag{5.43}$$

where $\tilde{H}_p$ is a smooth function of the four real variables, and $\bar{\alpha}$ is sufficiently smooth function.

At the time of maturity T we have

$$\tilde{H}_p(T, r, v, T) = 1 \quad \text{for all } v \text{ and } r.$$

In this case moves in the zero-coupon bond are described by the discounted GOP $\bar{V}_t^{\phi_*}$ and the short rate r_t. Now, from Assumption 5.1 and the Itô formula we get the following price dynamics for the zero-coupon bond.

Proposition 5.1 *Suppose the zero-coupon bond price satisfies Assumption (5.1). Then $\tilde{H}_p$ satisfies the term structure equation*

$$\begin{cases} \dfrac{\partial \tilde{H}_p}{\partial t} + \mu_t \dfrac{\partial \tilde{H}_p}{\partial r} + \dfrac{1}{2}\displaystyle\sum_{k=1}^{d} \left(\beta_t^k\right)^2 \dfrac{\partial^2 \tilde{H}_p}{\partial r^2} + \dfrac{1}{2}\alpha_t v \dfrac{\partial^2 \tilde{H}_p}{\partial v^2} + \dfrac{1}{2} v \displaystyle\sum_{k=1}^{d} \theta_t^k \beta_t^k \dfrac{\partial^2 \tilde{H}_p}{\partial v \partial r} = r\tilde{H}_p \\ \\ \tilde{H}_p(T, r, v, T) = 1 \end{cases} \tag{5.44}$$

Proof. First, we note that the GOP dynamics of can be rewritten as

$$d\bar{V}_t^{\phi_*} = \alpha_t dt + \bar{V}_t^{\phi_*} \sum_{k=1}^{d} \theta_t^k dW_t^k. \tag{5.45}$$

Applying Itô formula to (5.42) and using GOP dynamics (5.45) and short rate dynamics (5.41) we get the dynamics of the zero-coupon bond price under the real world probability P as follows

$$\begin{aligned} dp(t,T) &= \left\{ \frac{\partial \tilde{H}_p}{\partial t} + \alpha_t \frac{\partial \tilde{H}_p}{\partial v} + \left[\mu_t + \sum_{k=1}^{d} \beta_t^k \theta_t^k \right] \frac{\partial \tilde{H}_p}{\partial r} + \frac{1}{2}\alpha_t \bar{V}_t^{\phi_*} \frac{\partial^2 \tilde{H}_p}{\partial v^2} + \frac{1}{2}\sum_{k=1}^{d} (\beta_t^k)^2 \frac{\partial^2 \tilde{H}_p}{\partial r^2} \right. \\ &\quad \left. + \frac{1}{2}\bar{V}_t^{\phi_*} \sum_{k=1}^{d} \theta_t^k \beta_t^k \frac{\partial^2 \tilde{H}_p}{\partial v \partial r} \right\} dt + \sum_{k=1}^{d} \left[\bar{V}_t^{\phi_*} \theta_t^k \frac{\partial \tilde{H}_p}{\partial v} + \beta_t^k \frac{\partial \tilde{H}_p}{\partial r} \right] dW_t^k. \end{aligned} \tag{5.46}$$

On the other hand we have

$$\begin{aligned} d\hat{p}(t,T) &= d\left(\frac{p(t,T)}{V_t^{\phi_*}} \right) = d\left(\frac{\tilde{H}_p(t, r_t, \bar{V}_t^{\phi_*}, T)}{V_t^{\phi_*}} \right) \\ &= \tilde{H}_p(t, r_t, \bar{V}_t^{\phi_*}, T) d\left(\frac{1}{V_t^{\phi_*}} \right) + \frac{1}{V_t^{\phi_*}} d\tilde{H}_p(t, r_t, \bar{V}_t^{\phi_*}, T) \\ &\quad + \left\langle \tilde{H}_p(\cdot, r_\cdot, \bar{V}_\cdot^{\phi_*}, T), \frac{1}{V_\cdot^{\phi_*}} \right\rangle_t. \end{aligned}$$

and

$$d\left(\frac{1}{V_t^{\phi_*}} \right) = -\frac{1}{V_t^{\phi_*}} \left[r_t dt + \sum_{k=1}^{d} \theta_t^k dW_t^k \right]$$

where $\langle \cdot, \cdot \rangle_t$ denotes the quadratic variation. Hence, the benchmarked zero-coupon bond dynamics become

$$\begin{aligned} d\hat{p}(t,T) &= \frac{1}{V_t^{\phi_*}} \left\{ -r_t \tilde{H}_p + \frac{\partial \tilde{H}_p}{\partial t} + \mu_t \frac{\partial \tilde{H}_p}{\partial r} + \frac{1}{2} \sum_{k=1}^{d} \left(\beta_t^k\right)^2 \frac{\partial^2 \tilde{H}_p}{\partial r^2} + \frac{1}{2} \alpha_t \bar{V}_t^{\phi_*} \frac{\partial^2 \tilde{H}_p}{\partial v^2} \right. \\ & \left. + \frac{1}{2} \bar{V}_t^{\phi_*} \sum_{k=1}^{d} \theta_t^k \beta_t^k \frac{\partial^2 \tilde{H}_p}{\partial v \partial r} \right\} dt + \frac{1}{V_t^{\phi_*}} \sum_{k=1}^{d} \left[-\tilde{H}_p \theta_t^k + \frac{\partial H_p}{\partial v} \bar{V}_t^{\phi_*} \theta_t^k + \beta_t^k \frac{\partial \tilde{H}_p}{\partial r} \right] dW_t^k \end{aligned}$$

and the result follows from the fact that the benchmarked zero-coupon bond with maturity T is martingale under the real world probability measure P, so its drift term must be zero. ■

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