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An Extension of Ausubel's Auction for Heterogeneous Discrete Goods

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Abstract

Ausubel's [2] dynamic private-values auction for heterogeneous discrete goods yields an efficient equilibrium outcome but it works in a limited environment. If bidders' values for bundles are not integers, then the auction mechanism may not yield an efficient allocation without any information on bidders' values. In this paper, I extend Ausubel's auction for heterogeneous discrete goods to real-valued quasilinear utility functions. The mechanism I propose reaches a Walrasian equilibrium price vector in finite "steps" without any additional information on bidders' values. I identify prices reached at each step. In the extension of Ausubel's auction, truthful bidding constitutes an efficient equilibrium.

JEL Category D44
Keywords: Auctions, Ausubel auction, heterogeneous goods auction, price adjustment, tâtonnement.

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1 Introduction

Auctioning of multiple goods has become a rapidly developing part of the auction theory since the Federal Communications Commission began auctioning wireless communication bands in 1994. It is well-known that when many units of a good are to be auctioned, standard auction mechanisms are generally inefficient, i.e., they do not award goods to buyers who value them most. This inefficiency arises from the demand-reduction problem (see, for example, Krishna [8], Ausubel and Cramton [3], and Milgrom [9]). In the demand-reduction problem, buyers tend to shade their bids, resulting in an inefficient allocation of goods. Vickrey's sealed-bid auction, Vickrey [10], is among the few exceptions immune to this problem. Even though the Vickrey auction is efficient, it is not widely used in practice because bidders are supposed to submit their whole demand curves. Ausubel [2] introduced an elegant dynamic auction, both for divisible and discrete goods and it is immune to the demand-reduction problem. However, Ausubel's auction for heterogeneous discrete goods works in a limited environment. Bidders' values for bundles are restricted to be integers. The reason for this integer restriction is not explained in his paper. Moreover, the price adjustment procedure (the global Walrasian tâtonnement, see Ausubel [1]) is specifically designed to use integer property of utility functions. If bidders' values for bundles are not integers, without information on these values, the global Walrasian tâtonnement may not converge to a Walrasian equilibrium price vector.

In this paper, I extend Ausubel's auction for heterogeneous discrete goods to real-valued quasilinear utility functions by using an extension of the global Walrasian tâtonnement. I show that the extended Ausubel auction for heterogeneous discrete goods has an efficient equilibrium and yields Walrasian equilibrium prices when bidders' values for bundles are real numbers. I extend Ausubel's ascending and descending price adjustment procedures, the main components of the global Walrasian tâtonnement, to real-valued quasilinear utility functions. I show that the extended ascending and the extended descending procedures (the EAPAP and the EDPAP) converge to a Walrasian equilibrium price vector in finite "steps". Using these extended procedures, I extend the global Walrasian tâtonnement to real-valued quasilinear utility functions and show its convergence to a Walrasian equilibrium price vector in finite "steps". Unlike the global Walrasian tâtonnement of Ausubel [2], in the extended global Walrasian tâtonnement, the auctioneer does not need any information on the values bidders' utility functions take when these values are not integers. Some of the theorems and proofs are analogous to those in Ausubel [1].

In Ausubel's [2] auction bidders submit their demands as prices are adjusted. A bidder is credited an object if the rest of the bidders lower their demand for this good, and the bidder pays the current price for this object. An object is debited from a bidder at the current price if the rest of the bidders increase their demand for this good. The auctioneer calculates the set of goods in excess demand and adjusts the prices accordingly. The auction ends whenever there is a market clearing allocation demanded by bidders at the current price. Bidders are assumed to have private values for objects (each bidder's values for objects depend only on his own type) and have utility functions quasilinear in money. In the case of divisible goods, bidders have concave utility functions whereas in the case of discrete goods, they have preferences satisfying the *gross substitutes* condition. The gross substitutes assumption basically requires that a bidder's demand for a good to be nondecreasing if its price remain the same while the rest of the prices do not decrease. This assumption guarantees the existence of a Walrasian equilibrium (see, for example, Gul and Stachetti [6]). When goods are divisible, the classical Walrasian tâtonnement price adjustment procedure is used to determine the path of prices. In the case of discrete goods, the global Walrasian tâtonnement (see Ausubel [1]) is used. Ausubel [1] shows that the global Walrasian tâtonnement converges to a Walrasian equilibrium price vector in finite *steps*. Moreover, if bidders report their demands truthfully, then the auctioneer can extend this discrete-time price adjustment procedure to a continuous-time price adjustment procedure due to the integer restriction on utility functions, Ausubel [2]. The auctioneer achieves this extension by linearly increasing (or decreasing) prices between two consecutive integer-valued price vectors determined by the global Walrasian tâtonnement. Ausubel [2] uses this continuous-time price adjustment procedure to prove that sincere bidding by bidders comprise an efficient equilibrium and yields Walrasian equilibrium prices. Moreover, a procedure with parallel auctions is introduced and shown to yield Vickrey-Clarke-Groves (VCG) payoffs (Vickrey [10], Clarke [4], Groves [5]) to bidders starting from any integer-valued initial price vector if they bid sincerely. Ausubel's auction is privacy-preserving and simpler than the Vickrey auction. In the Vickrey auction, bidders need to submit their whole demand curves. Ausubel [2] shows that even though bidders have market power, nonlinear pricing feature of the auction solves the demand-reduction problem. This is because strategic bidders behave as price-takers under this nonlinear pricing rule.

In the extension of Ausubel's auction for discrete heterogeneous goods, the auction starts with an initial price vector $\boldsymbol{p}(0)$. Each bidder submits his report, a set of bundles he demands at $\boldsymbol{p}(0)$. Prices are adjusted continuously according to the extended global Walrasian tâtonnement. Bidders submit their reports whenever a new bundle is added to these reports as prices change. At any time $t \in [0, \infty)$, if there is a bidder who submits a new report at the current price $\boldsymbol{p}(t)$, then the price adjustment stops. For any bidder, $i \in N$, if opponents of bidder i lower their demand for a good, then the good is credited to bidder i at price $\boldsymbol{p}(t)$. On the other hand, if opponents of bidder i rise their demand for a good, then it is debited from bidder i at price $\boldsymbol{p}(t)$. After crediting and debiting of goods are over, prices are adjusted continuously according to the extended global Walrasian tâtonnement. The auction ends at time $T \in [0, \infty)$ when there is a market clearing allocation of objects in these reports made at time T. Finally, payments and good transfers are made.

Section 2 gives the assumptions of the model. In Section 3, the extension of Ausubel's auction and the extended global Walrasian tâtonnement are explained. Section 4 explains how to identify paths of the EAPAP and the EDPAP, and shows convergence of the EAPAP, the EDPAP, and the extended global Walrasian tâtonnement in finite steps to Walrasian equilibrium price vectors. Section 5 gives the main results.

2 The Model

Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for the sets of integer, rational and real numbers respectively. There are finite number of goods, $K = \{1, 2, \ldots, K\}$. There is a seller with supply $\boldsymbol{S} = (S^k)_{k \in K} \in \mathbb{Z}^K_{++}$ of discrete heterogeneous goods, and she wants to sell them to a finite group of bidders, $N = \{1, 2, \ldots, N\}$. Consumption set of a bidder $i \in N$ is $X_i = \{\boldsymbol{x} \in \mathbb{Z}^K : 0 \leq x^k \leq \overline{x}^k \text{ for all } k \in K\}$, and $\boldsymbol{x_i} = (x^k_i)_{k \in K} \in X_i$ is a bundle i consumes.

The following assumptions are made for each bidder i:

A.1 *Private Values:* Bidder i's utility function $u_i : X_i \times \mathbb{R} \to \mathbb{R}$ is a function of bundle $\boldsymbol{x_i} \in X_i$ and money $t_i \in \mathbb{R}$ he consumes, and it does not depend on any information about other bidders.

A.2 *Quasilinearity:* $u_i(\cdot)$ is assumed to be quasilinear in money, i.e., there exists $U_i : X_i \to \mathbb{R}$ such that for each $x_i \in X_i$ and each $t_i \in \mathbb{R}$,

$$u_i(\boldsymbol{x_i}, t_i) = U_i(\boldsymbol{x_i}) + t_i$$

where $U_i(\boldsymbol{x_i})$ is i's value for bundle $\boldsymbol{x_i}$.

Initial wealth $m_i \in \mathbb{R}_{++}$ of each bidder $i \in N$ is so large that his budget constraint does not bind for any bundle he demands at any price. Since

$u_i(\boldsymbol{x_i}, m_i - y_i) = U_i(\boldsymbol{x_i}) + m_i - y_i$ where y_i is the amount of expense he makes, m_i will be dropped without loss of generality.

A.**3** *Strict Monotonicity:* For all $(\boldsymbol{x'_i}, t'_i), (\boldsymbol{x_i}, t_i) \in X_i \times \mathbb{R}$ such that $(\boldsymbol{x'_i}, t'_i) \gneqq (\boldsymbol{x_i}, t_i)$

$$u_i(\boldsymbol{x'_i}, t'_i) > u_i(\boldsymbol{x_i}, t_i).$$

Bidder i's *indirect utility function* at price vector $\boldsymbol{p} = (p^k)_{k \in K} \in \mathbb{R}^K_+$ is

$$V_i(\boldsymbol{p}) = \max_{\boldsymbol{x_i} \in X_i} \{U_i(\boldsymbol{x_i}) - \boldsymbol{p} \cdot \boldsymbol{x_i}\},$$

and his *demand correspondence* (*demand set*) at price vector $\boldsymbol{p} \in \mathbb{R}^K_+$ is

$$Q_i(\boldsymbol{p}) = \arg\max_{\boldsymbol{x_i} \in X_i} \{U_i(\boldsymbol{x_i}) - \boldsymbol{p} \cdot \boldsymbol{x_i}\}.$$

In the following assumption, each good is assumed to be available in unit supply. This version of the gross substitutes assumption is sufficient and "almost necessary" for the existence of the Walrasian equilibrium. The mechanism proposed in this paper, like Ausubel's [2] auction, allows multiple units of each good without loss of generality.

A.**4** *Gross Substitutes:* For all price vectors $\boldsymbol{p}, \boldsymbol{p'} \in \mathbb{R}^K_+$ such that $\boldsymbol{p'} \geq \boldsymbol{p}$, if demand $Q_i(\cdot)$ is single-valued both at $\boldsymbol{p}$ and at $\boldsymbol{p'}$, $\boldsymbol{x_i} \in Q_i(\boldsymbol{p})$, and $\boldsymbol{x'_i} \in Q_i(\boldsymbol{p'})$, then $x'^k_i \geq x^k_i$ for each $k \in K$ such that $p'^k = p^k$.

A *Walrasian equilibrium* is $(\boldsymbol{p^*}, \boldsymbol{x^*})$, $\boldsymbol{p^*}$ is equilibrium price vector and $\boldsymbol{x^*} = (\boldsymbol{x_i})_{i \in N}$ is equilibrium allocation such that for each bidder $i \in N$ $\boldsymbol{x^*_i} \in Q_i(\boldsymbol{p^*})$, and $\sum_{i \in N} \boldsymbol{x^*_i} = \boldsymbol{S}$.

In Ausubel's [2] auction, bidders' values for bundles are assumed to be integer. The price adjustment procedure (the global Walrasian tâtonnement) in Ausubel [2] is designed to use this property of utility functions to move on the grid of integer price vectors integer prices, and to reach an integer-valued Walrasian equilibrium price vector. If bidders' values for bundles are not integer then there may not exist an integer-valued Walrasian equilibrium price vector, and the price adjustment procedure may not yield an efficient allocation when it ends. In order to reach a Walrasian equilibrium price vector, the price adjustment procedure needs more information on bidders' values. On the other hand, as Lemma 1 below shows, relaxing the integer-values restriction enriches the class of preferences the auction works.

Lemma 1 shows that the class of environments on which the extensions of the price adjustment procedures and the extension of Ausubel's auction given in this paper are richer than those in Ausubel [2].

A preference relation $\mathcal{R}$ defined on $X \times \mathbb{R}$ is *representable* if there exists a utility function $u : X \times \mathbb{R} \to \mathbb{R}$ such that for each $\boldsymbol{x}, \boldsymbol{y} \in X \times \mathbb{R}$,

$$\boldsymbol{x} \mathcal{R} \boldsymbol{y} \text{ if and only if } u(\boldsymbol{x}) \geq u(\boldsymbol{y}).$$

A preference relation $\mathcal{R}$ is said to be in $\mathcal{R}_D$ if and only if there exists a quasilinear utility function $u : X \times \mathbb{R} \to \mathbb{R}$ representing $\mathcal{R}$ such that $u(\cdot, t) = U(\cdot) + t$ where $U : X \to D$.

Lemma 1 shows that sets $\mathcal{R}_{\mathbb{Z}}$, and $\mathcal{R}_{\mathbb{R}}$ are different.

Lemma 1. *$\mathcal{R}_{\mathbb{Z}} \subsetneq \mathcal{R}_{\mathbb{R}}$.*

Proof. $\mathcal{R}_{\mathbb{Z}} \subset \mathcal{R}_{\mathbb{R}}$ is trivial. I will show that there exists $\mathcal{R} \in \mathcal{R}_{\mathbb{R}}$ such that $\mathcal{R} \notin \mathcal{R}_{\mathbb{Z}}$. Let $\mathcal{R} \in \mathcal{R}_{\mathbb{R}}$ such that there exist $\boldsymbol{x}, \boldsymbol{y} \in X$ and $\bar{t} \in \mathbb{R} \setminus \mathbb{Z}$ such that

$$(\boldsymbol{x}, \bar{t}) \underset{\mathcal{R}}{\backsim} (\boldsymbol{y}, 0). \tag{1}$$

By definition of $\mathcal{R}_{\mathbb{R}}$, there exists $u(\cdot, t) = U(\cdot) + t$ representing $\mathcal{R}$, where $U : X \to \mathbb{R}$. Equation 1 implies that

$$u(\boldsymbol{x}, \bar{t}) = u(\boldsymbol{y}, 0).$$

Therefore,

$$\bar{t} = U(\boldsymbol{y}) - U(\boldsymbol{x}) \in \mathbb{R} \setminus \mathbb{Z}. \tag{2}$$

Now, assume on the contrary that, there exists $\widetilde{u}(\cdot, t) = \widetilde{U}(\cdot) + t$ representing $\mathcal{R}$ such that

$$\widetilde{U} : X \to \mathbb{Z}. \tag{3}$$

Equation 1 implies that $\widetilde{u}(\boldsymbol{x}, \bar{t}) = \widetilde{u}(\boldsymbol{y}, 0)$. So, $\bar{t} = \widetilde{U}(\boldsymbol{y}) - \widetilde{U}(\boldsymbol{x})$. Equation 2 implies that $\bar{t} = \widetilde{U}(\boldsymbol{y}) - \widetilde{U}(\boldsymbol{x}) \in \mathbb{R} \setminus \mathbb{Z}$, a contradiction to equation 3. Hence, $\mathcal{R}_{\mathbb{Z}} \subsetneq \mathcal{R}_{\mathbb{R}}$. ☐

3 The Extension of Ausubel's Auction

In the extension of Ausubel's auction for discrete heterogeneous goods, there are N bidders and an auctioneer with supply $\boldsymbol{S} \in \mathbb{Z}^K_{++}$ of goods to sell. The auction starts at an arbitrary initial price vector $\boldsymbol{p}(0) \in \mathbb{R}^K_+$, and the auctioneer adjusts prices continuously according to the extended global Walrasian tâtonnement. The extended global Walrasian tâtonnement has two components: the EAPAP, and the EDPAP. The extended global Walrasian tâtonnement starts with the EAPAP at an arbitrary price vector $\boldsymbol{p}(0) \in \mathbb{R}^K_+$.

The EAPAP runs until it terminates. The price vector at which the EAPAP terminated is used as the starting price vector for the EDPAP. The EDPAP runs until it terminates. The price vector at which the EDPAP terminated is used as the starting price vector for the EAPAP. The EAPAP runs until it terminates, and so on. These steps are repeated until the price vector reached no longer changes in both the EAPAP and the EDPAP. When the EAPAP starts at $\boldsymbol{p}(0)$, each bidder $i \in N$ reports his demand set $x_i(\boldsymbol{p}(0)) \subset X_i$ at $\boldsymbol{p}(0)$. The auctioneer determines the set of goods in excess demand $E_+(\boldsymbol{p}(0))$ and their prices are increased at the same rate continuously whereas prices of the rest of the goods remain the same. As prices are being adjusted, bidders may add bundles to their demand set or they may remove bundles from their demand set. Each bidder is required to report his demand set at all price vectors at which he adds a bundle to his demand set. Time $t \in [0, \infty)$ of the price adjustment is called a *step* of the procedure if there is a bidder who reports his demand set at $\boldsymbol{p}(t)$. If a bidder reports his demand set, then the auctioneer stops the price adjustment, and determines demand sets of the rest of the bidders using their latest reported demand sets. Then, the auctioneer computes the set of goods in excess demand $E_+(\boldsymbol{p}(t))$. Next, the auctioneer resumes the price adjustment by increasing prices of all goods in $E_+(\boldsymbol{p}(t))$ at the same rate continuously, and holds the prices of the rest of the goods the same. The EAPAP terminates when the set of goods in excess demand is an empty set. The EDPAP works analogously. The price adjustment can be written as

$$\frac{\mathrm{d}p^k(t)}{\mathrm{d}t} = \begin{cases} c_{E(\boldsymbol{p}(t))} & \text{if } k \in E(\boldsymbol{p}(t)) \\ 0 & \text{if } k \notin E(\boldsymbol{p}(t)) \end{cases} \tag{4}$$

where

$$E(\boldsymbol{p}(t)) = E_+(\boldsymbol{p}(t)) \text{ and } c_{E_+(\boldsymbol{p}(t))} > 0$$

when the price adjustment is the EAPAP, and

$$E(\boldsymbol{p}(t)) = E_-(\boldsymbol{p}(t)) \text{ and } c_{E_-(\boldsymbol{p}(t))} < 0$$

when the price adjustment is the EDPAP.

At each *step* t, for each bidder j, the auctioneer determines bid $\boldsymbol{x_j}(t) \in x_j(\boldsymbol{p}(t))$ of bidder j at $\boldsymbol{p}(t)$. For each bidder $i \in N$, if opponents of bidder i lower the quantity they demand of a good in their bids, then it is credited to bidder i at price $\boldsymbol{p}(t)$. On the other hand, if opponents of bidder i rise the quantity they demand of a good, then it is debited from bidder i at price $\boldsymbol{p}(t)$. After crediting and debiting of goods are calculated, prices are continued to be adjusted according to the extended global Walrasian tâtonnement. The auction ends, say at time $T \in [0, \infty)$, whenever there is a market clearing allocation such that each bidder receives a bundle from his demand set at $\boldsymbol{p}(T)$. Finally, for each bidder, credits and debits are added, monetary transfers are made, and goods are allocated. For an illustration of Ausubel's auction, see the example in Ausubel [2], pp. 606-607.

For each $i \in N$, payment of bidder i is computed by

$$a_i(T) = \boldsymbol{p}(0) \cdot [\boldsymbol{S} - \boldsymbol{x_{-i}}(0)] - \int_0^T \boldsymbol{p}(t) \cdot \mathrm{d}\boldsymbol{x_{-i}}(t)$$

where $\boldsymbol{x_i}(t)$ is a bundle bidder i demands at price $\boldsymbol{p}(t)$ and $\boldsymbol{x_{-i}}(t) = \sum_{j \neq i, j \in N} \boldsymbol{x_i}(t)$.

Ausubel [2] (Lemma 1) shows that if $\boldsymbol{p}(\cdot)$ is continuous and for each $j \in N, j \neq i$, any $k \in K$ $x_j^k(\cdot)$, amount of good k demanded by bidder j, is of bounded variation, then $a_i(T)$ is well-defined. Moreover, if $\boldsymbol{p}(\cdot)$ is also a piecewise smooth function from $[0, T]$ to $\mathbb{R}^K$, and for each $j \in N, j \neq i$, $U_j(\cdot)$ is a concave and continuous function, then by Lemma 2 in Ausubel [2] to

$$a_i(T) = \boldsymbol{p}(0) \cdot [\boldsymbol{S} - \boldsymbol{x_{-i}}(0)] - \sum_{j \neq i, j \in N} [U_j(\boldsymbol{x_j}(\boldsymbol{p}(T))) - U_j(\boldsymbol{x_j}(\boldsymbol{p}(0)))].$$

For a detailed discussion of this payment function see Ausubel [2].

In the global Walrasian tâtonnement of Ausubel [2] (and Ausubel [1]), prices are adjusted in discrete time and they take integer values whereas in the extended global Walrasian tâtonnement, they are adjusted continuously. Ausubel [2] shows that the global Walrasian tâtonnement can be made a continuous procedure by linearly increasing prices between consecutive integer-valued price vectors. The extended global Walrasian tâtonnement and the continuous version of the global Walrasian tâtonnement follow the same path if bidders' values for bundles are integer, the initial price vector is integer-valued, and the EAPAP has a unit rate of change.

4 The Extended Ascending and Descending Price Adjustment Procedures

4.1 Determining the Price Path

In the EAPAP and the EDPAP, at each price $\boldsymbol{p}$, given the demand reports of bidders, the set of goods in excess demand is found using the function

$$L(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{S} + \sum_{i \in N} V_i(\boldsymbol{p}). \tag{5}$$

where $L : \mathbb{R}_+^K \to \mathbb{R}$ is a *Lyapunov function.* Note that this function, by envelope theorem, is minimized at Walrasian equilibrium prices. The following Lemma from Ausubel [1] shows important properties of the Lyapunov function $L(\cdot)$ of equation 5.

A function $L : \mathbb{R}_+^K \to \mathbb{R}$ is *submodular* if for each $\boldsymbol{p}, \boldsymbol{p'} \in \mathbb{R}_+^K$

$$L(\boldsymbol{p} \wedge L(\boldsymbol{p'}) + L(L(\boldsymbol{p} \vee L(\boldsymbol{p'}) \leq L(\boldsymbol{p}) + L(\boldsymbol{p'}).$$

Lemma 2 (Ausubel [1])**.** *Under the* gross substitutes *assumption, A4, the Lyapunov function $L(\cdot)$ of equation 5 is a submodular and a convex function.*

Proposition 1 below is from Ausubel [1] and is still valid without any change when bidders' values for bundles are real rather than integer. Proposition 1 shows the relationship between the Walrasian equilibrium, the Lyapunov function, and the social surplus.

Proposition 1 (Ausubel [1])**.** *Suppose that assumptions $A1 - A3$ hold, and that a Walrasian equilibrium exists. Then, the set of Walrasian equilibrium price vectors equals the set of minimizers of $L(\cdot)$, and the set of Walrasian equilibria equals the set of all $(\boldsymbol{p^*}, \boldsymbol{x^*})$ such that $\boldsymbol{p^*} \in \mathbb{R}_+^K$ minimizes $L(\cdot)$ and $(\boldsymbol{x_i^*})_{i \in N}$ maximizes*

$$\sum_{i \in N} U_i(\boldsymbol{x_i}) \textit{ subject to } \boldsymbol{x_i} \in X_i \textit{ for all } i \in N,$$

and

$$\sum_{i \in N} \boldsymbol{x_i} \leq \boldsymbol{S}.$$

The following *Corollary to Proposition 1* is also from Ausubel [1], but integer properties of the highest and the lowest Walrasian price vectors are dropped as they are not necessarily true when bidders' values for bundles are not restricted to integers. For the proof of the *Corollary to Proposition 1*, see Ausubel [1].

Corollary 1. *Suppose that assumptions* $A1 - A4$ *hold. Then, the set of Walrasian equilibrium price vectors is a nonempty lattice, and there exist the lowest and the highest Walrasian equilibrium price vectors,* $\underline{\boldsymbol{p}} \in \mathbb{R}_+^K$ *and* $\overline{\boldsymbol{p}} \in \mathbb{R}_+^K$*, respectively.*

Corollary 1 implies that for any economy $(\{u_i(\cdot)\}_{i\in N}, \boldsymbol{S})$, there exist Walrasian equilibrium price vectors $\underline{\boldsymbol{p}} \in \mathbb{R}_+^K$ and $\overline{\boldsymbol{p}} \in \mathbb{R}_+^K$ such that if $\boldsymbol{p^*} \in \mathbb{R}_+^K$ is a Walrasian equilibrium price vector, then $\underline{\boldsymbol{p}} \leq \boldsymbol{p^*} \leq \overline{\boldsymbol{p}}$.

Suppose that a bidder, $i \in N$, announces his demand set at $Q_i(\boldsymbol{p}) \subset X_i$ at price vector $\boldsymbol{p} \in \mathbb{R}_+^K$. I will now show how the auctioneer can determine bundles that will stay in i's demand set when a subset of prices are all slightly increased (or decreased).

For each $\delta \in \mathbb{R}$, and for each $E \subset K$, let $\boldsymbol{\delta}^E = (\delta^k)_{k\in K}$ denote a K-dimensional vector such that $\delta^k = 0$ if $k \notin E$, and $\delta^k = \delta$ if $k \in E$. For each $\boldsymbol{p} \in \mathbb{R}_+^K$, and for each $\boldsymbol{\Delta} \in \{\boldsymbol{\Delta'} : \boldsymbol{0} \leq \boldsymbol{\Delta'} \leq \boldsymbol{\delta_i}^K\}$, a *minimal cost increase bundle*, $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p})$, when prices increase from $\boldsymbol{p}$ to $\boldsymbol{p} + \boldsymbol{\Delta}$, is defined as

$$\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in argmin_{x\in Q_i(\boldsymbol{p})}\{\boldsymbol{\Delta} \cdot x\}. \tag{6}$$

Analogously, a *maximal cost decrease bundle*, $\widetilde{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p})$, when prices decrease from $\boldsymbol{p}$ to $\boldsymbol{p} - \boldsymbol{\Delta}$ is defined as

$$\widetilde{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in argmax_{x\in Q_i(\boldsymbol{p})}\{\boldsymbol{\Delta} \cdot x\}.$$

For each $i \in N$, and for each $\boldsymbol{p} \in \mathbb{R}_+^K$, define

$$F_i(\boldsymbol{p}) = \{(x, y) \in X_i \times X_i | U_i(x) - \boldsymbol{p} \cdot x \neq U_i(y) - \boldsymbol{p} \cdot y\},$$

and construct $\delta_i(\boldsymbol{p})$ as follows:

$$\delta_i(\boldsymbol{p}) = \begin{cases} min_{(x,y)\in F_i(\boldsymbol{p})} |(U_i(x) - \boldsymbol{p} \cdot x) - (U_i(y) - \boldsymbol{p} \cdot y)| & \text{if } F_i(\boldsymbol{p}) \neq \emptyset \\ 1 & \text{if } F_i(\boldsymbol{p}) = \emptyset. \end{cases} \tag{7}$$

Note that $\delta_i(\cdot)$ is well-defined because $X_i \subset \mathbb{Z}$ is bounded for each $i \in N$.

Proposition 2. *Suppose that assumptions* $A1 - A4$ *hold. Then, for each bidder* $i \in N$*, and for each price vector* $\boldsymbol{p} \in \mathbb{R}_+^K$*, there exists* $\delta_i(\boldsymbol{p}) \in \mathbb{R}_{++}$

such that for each $\boldsymbol{\Delta} \in \mathbb{R}_+^K$ *such that* $\mathbf{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta_i}(\boldsymbol{p})^K$*, and for each* $\lambda \in \mathbb{R}_{++}$ *such that* $\mathbf{0} \leq \lambda\boldsymbol{\Delta} \leq \boldsymbol{\delta_i}(\boldsymbol{p})^K$*,*

$$\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \text{ and } \widetilde{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p} - \lambda\boldsymbol{\Delta}). \tag{8}$$

Proof. I will show that $\delta_i(\boldsymbol{p})$ satisfies the statement of the Proposition. There are two cases:

Case 1 $F_i(\boldsymbol{p}) \neq \emptyset$: Suppose, on the contrary that, there exist $\boldsymbol{\Delta} \in \mathbb{R}_+^K$ and $\lambda \in \mathbb{R}_{++}$ such that $\mathbf{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta_i}(\boldsymbol{p})^K$, $\mathbf{0} \leq \lambda\boldsymbol{\Delta} \leq \boldsymbol{\delta_i}(\boldsymbol{p})^K$, and

$$\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \notin Q_i(\boldsymbol{p} + \lambda\boldsymbol{\Delta}). \tag{9}$$

Gul and Stacchetti [6] (Lemma 2) showed that the gross substitutes preferences also satisfy the *single-improvement* property: if $\boldsymbol{x_i} \notin Q_i(\boldsymbol{p})$, then there exists $\boldsymbol{x'_i} \in X_i$ such that $\#(\boldsymbol{x_i} \setminus \boldsymbol{x'_i}) \leq 1$, $\#(\boldsymbol{x'_i} \setminus \boldsymbol{x_i}) \leq 1$, and $U_i(\boldsymbol{x'_i}) - \boldsymbol{p} \cdot \boldsymbol{x'_i} > U_i(\boldsymbol{x_i}) - \boldsymbol{p} \cdot \boldsymbol{x_i}$. Therefore, equation 9 implies that there exists $\boldsymbol{x'_i} \in X_i$ such that

$$\#(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \setminus \boldsymbol{x'_i}) \leq 1 \text{ and } \#(\boldsymbol{x'_i} \setminus \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) \leq 1,$$

and

$$U_i(\boldsymbol{x'_i}) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \boldsymbol{x'_i} > U_i(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}). \tag{10}$$

The definition of $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\cdot)$, equation 6, implies that for each $\boldsymbol{y} \in Q_i(\boldsymbol{p})$,

$$U_i(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \geq U_i(y) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \boldsymbol{y} \tag{11}$$

Therefore, by inequality 10,

$$\boldsymbol{x'_i} \notin Q_i(\boldsymbol{p}), \tag{12}$$

$$(\boldsymbol{x'_i}, \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) \in F_i(\boldsymbol{p}) \tag{13}$$

and

$$\delta_i(\boldsymbol{p}) > 0. \tag{14}$$

Furthermore, by Definition of $\delta_i(\cdot)$ in equation 7

$$U_i(\boldsymbol{x'_i}) - \boldsymbol{p} \cdot \boldsymbol{x'_i} \leq U_i(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) - \boldsymbol{p} \cdot \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) - \delta_i(\boldsymbol{p}). \tag{15}$$

Moreover, for each $\lambda \in \mathbb{R}_{++}$ such that $\mathbf{0} \leq \lambda\boldsymbol{\Delta} \leq \boldsymbol{\delta_i}(\boldsymbol{p})^K$, if

$$\#(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \setminus \boldsymbol{x'_i}) \leq 1,$$

then

$$\lambda \boldsymbol{\Delta} \cdot \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \leq \lambda \boldsymbol{\Delta} \cdot \boldsymbol{x}'_{\boldsymbol{i}} + \delta_i(\boldsymbol{p}). \tag{16}$$

Inequalities 15 and 16 imply that

$$U_i(\boldsymbol{x}'_{\boldsymbol{i}}) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \boldsymbol{x}'_{\boldsymbol{i}} \leq U_i(\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})) - (\boldsymbol{p} + \lambda\boldsymbol{\Delta}) \cdot \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}),$$

a contradiction to inequality 10. So, $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p} + \lambda\boldsymbol{\Delta})$.

Case 2 $F_i(\boldsymbol{p}) = \emptyset$: $F_i(\boldsymbol{p}) = \emptyset$ means that bidder i is indifferent between all bundles at price $\boldsymbol{p}$. By inequality 11,

$$\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p} + \lambda\boldsymbol{\Delta}).$$

Analogously it can be shown that $\widetilde{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta}) \in Q_i(\boldsymbol{p} - \lambda\boldsymbol{\Delta})$. $\square$

For each $\boldsymbol{p} \in \mathbb{R}^K_+$, define

$$\delta(\boldsymbol{p}) = \min_{i \in N} \delta_i(\boldsymbol{p}). \tag{17}$$

Corollary 2. *Suppose that assumptions $A1 - A4$ hold, and that bidders truthfully report their demand. Then, for each step t of the EAPAP (and the EDPAP), if the procedure does not terminate at t, then there exists a step $t' > t$ such that there does not exists any step $\hat{t}$ such that $\hat{t} \in (t, t')$.*

Corollary 2 follows from Proposition 2, definition of $\delta(\cdot)$ above, and finiteness of N.

The next Proposition shows that for any price there exists a small neighborhood such that there exists a unique minimal set of goods which determine the direction in which prices increase (or decrease) in that neighborhood.

A *minimal minimizer* $\boldsymbol{p}_+$ is defined as

$$\boldsymbol{p}_+ \in \underset{\widetilde{\boldsymbol{p}} \in \{\boldsymbol{p}+\boldsymbol{\Delta}:\boldsymbol{0}\leq\boldsymbol{\Delta}\leq\boldsymbol{\delta}(\boldsymbol{p})^K\}}{\arg\min} \{L(\widetilde{\boldsymbol{p}})\}, \tag{18}$$

such that for each $\boldsymbol{p'} \in \mathbb{R}^K_+$ if $\boldsymbol{p} \leq \boldsymbol{p'} \lneq \boldsymbol{p}_+$, then $L(\boldsymbol{p'}) > L(\boldsymbol{p}_+)$.

So, a minimal minimizer $\boldsymbol{p}_+$ is a minimizer of the Lyapunov function of equation 17 in $\{\boldsymbol{p} + \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p})^K\}$ and this set does not contain any price vector which is less than or equal to $\boldsymbol{p}_+$ in every coordinate, and a minimizer of the Lyapunov function in that set.

Similarly, given $\boldsymbol{p} \in \mathbb{R}^K_+$, a *maximal minimizer* $\boldsymbol{p}_-$ is defined as

$$\boldsymbol{p}_- \in \underset{\widetilde{\boldsymbol{p}} \in \{\boldsymbol{p}-\boldsymbol{\Delta}:\boldsymbol{0}\leq\boldsymbol{\Delta}\leq\boldsymbol{\delta}(\boldsymbol{p})^K\}}{\arg\min} \{L(\widetilde{\boldsymbol{p}})\},$$

such that for each $\boldsymbol{p'} \in \mathbb{R}^K_+$ if $\boldsymbol{p} \geq \boldsymbol{p'} \gneq \boldsymbol{p}_-$, then $L(\boldsymbol{p'}) > L(\boldsymbol{p}_-)$.

Proposition 3. *Suppose that assumptions A1-A4 hold, and that bidders truthfully report their demand. Then, at each price* $\boldsymbol{p} \in \mathbb{R}_+^K$*, the minimal minimizer* $\boldsymbol{p}_+$ *and the maximal minimizer* $\boldsymbol{p}_-$ *are uniquely defined. Moreover, there exist* $E_+(\boldsymbol{p}), E_-(\boldsymbol{p}) \subset K$ *such that*

$$\boldsymbol{p}_+ = \boldsymbol{p} + \boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})} \tag{19}$$

and

$$\boldsymbol{p}_- = \boldsymbol{p} - \boldsymbol{\delta}(\boldsymbol{p})^{E_-(\boldsymbol{p})}.$$

Proof. As $L(\cdot)$ is continuous and $\{\boldsymbol{p} + \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p})^K\}$ is compact, $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$ exist.

By Lemma 2, $L(\cdot)$ is a submodular function, and so its minimizers on lattices $\{\boldsymbol{p} + \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p})^K\}$ and $\{\boldsymbol{p} - \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p})^K\}$ are nonempty sublattices. Therefore, there exist a unique minimal minimizer $\boldsymbol{p}_+$ and a unique maximal minimizer $\boldsymbol{p}_-$.

Now I will show that there exists $E_+(\boldsymbol{p})$ satisfying 19. Define

$$E = \{k \in K : p_+^k \notin \{p^k, p^k + \delta(\boldsymbol{p})\}\},$$

and

$$\forall k \in K \quad \widetilde{p}^k = \begin{cases} p^k & \text{if } k \in E \\ p_+^k & \text{if } k \notin E. \end{cases} \tag{20}$$

Let $\widetilde{\boldsymbol{\Delta}} = \boldsymbol{p}_+ - \widetilde{\boldsymbol{p}}$. Suppose on the contrary that $E \neq \emptyset$, i.e., there is no $E_+(\boldsymbol{p}) \subset K$ such that $\boldsymbol{p}_+ = \boldsymbol{p} + \boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})}$. Let $\overline{\lambda} = \delta(\boldsymbol{p})/max\{\widetilde{\boldsymbol{\Delta}}^k : k \in E\}$.

There are two cases:

Case 1 $L(\widetilde{\boldsymbol{p}} + \overline{\lambda}\widetilde{\boldsymbol{\Delta}}) \geq L(\widetilde{\boldsymbol{p}})$: As $L(\cdot)$ is convex, and since $\boldsymbol{p}_+$ is a convex combination of $\widetilde{\boldsymbol{p}} + \overline{\lambda}\widetilde{\boldsymbol{\Delta}}$ and $\widetilde{\boldsymbol{p}}$

$$L(\widetilde{\boldsymbol{p}}) \leq L(\boldsymbol{p}_+), \tag{21}$$

a contradiction to $\boldsymbol{p}_+$ being the unique a minimal minimizer.

Case 2 $L(\widetilde{\boldsymbol{p}} + \overline{\lambda}\widetilde{\boldsymbol{\Delta}}) < L(\widetilde{\boldsymbol{p}})$: As $L(\cdot)$ is convex, and since $\boldsymbol{p}_+$ is a convex combination of $\widetilde{\boldsymbol{p}} + \overline{\lambda}\widetilde{\boldsymbol{\Delta}}$ and $\widetilde{\boldsymbol{p}}$,

$$L(\widetilde{\boldsymbol{p}} + \overline{\lambda}\widetilde{\boldsymbol{\Delta}}) < L(\boldsymbol{p}_+), \tag{22}$$

a contradiction to $\boldsymbol{p}_+$ being a minimizer.

So, $E = \emptyset$ meaning that $\boldsymbol{p}_+ = \boldsymbol{p} + \boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})}$ for some $E_+(\boldsymbol{p}) \subset K$.

An analogous argument can be made for $E_-\boldsymbol{p}_-$.

□

The set of goods in excess demand $E_+(\boldsymbol{p})$ (and $E_-(\boldsymbol{p})$) determines the direction of the price path for the EAPAP (and the EDPAP). Observe that for each $E \subsetneq E_+(\boldsymbol{p})$,

$$L(\boldsymbol{p}+\boldsymbol{\delta}(\boldsymbol{p})^E) > L(\boldsymbol{p}+\boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})}) \tag{23}$$

by definition of $\boldsymbol{p_+}$. Similarly, for each $E \subsetneq E_-(\boldsymbol{p})$,

$$L(\boldsymbol{p}+\boldsymbol{\delta}(\boldsymbol{p})^E) > L(\boldsymbol{p}+\boldsymbol{\delta}(\boldsymbol{p})^{E_-(\boldsymbol{p})}) \tag{24}$$

by definition of $\boldsymbol{p_-}$.

The following Lemma gives a rather simple method for determining the set of goods in excess demand. This notion of excess demand is different from the classical notion of excess demand. Gul and Stachetti [7] point out to the distinction between the level of excess demand and the sum of quantities of goods in excess demand in discrete goods. For example, two different goods can be in excess demand but the level of excess demand may only be 1. The set of goods in excess demand according to the classical notion and the one here coincide if the demand correspondences are assumed to be single-valued. In otherwords, the same set of goods will be in excess demand.

Given $E \subset K$, define

$$\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, E) \in \operatorname*{arg\,min}_{\boldsymbol{x_i} \in Q_i(\boldsymbol{p})} \sum_{k \in E} x_i^k. \tag{25}$$

$\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, E)$ is a *minimal cost increase bundle* in $Q_i(\boldsymbol{p})$ when prices of goods in E are slightly increased at the same rate. Note that $\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, E) = \widetilde{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})$ if $\boldsymbol{\Delta} \in \{\boldsymbol{\Delta'} : \boldsymbol{0} \leq \boldsymbol{\Delta'} \leq \boldsymbol{\delta}(\boldsymbol{p})^E\}$.

Similarly, given $E \subset K$, define

$$\widehat{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, E) \in \operatorname*{arg\,max}_{\boldsymbol{x_i} \in Q_i(\boldsymbol{p})} \sum_{k \in E} x_i^k. \tag{26}$$

$\widehat{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, E)$ is a *maximal cost decrease bundle* in $Q_i(\boldsymbol{p})$ when prices of goods in E are slightly decreased at the same rate. Note that $\widehat{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, E) = \widetilde{\boldsymbol{y}}_{\boldsymbol{i}}(\boldsymbol{p}, \boldsymbol{\Delta})$ if $\boldsymbol{\Delta} \in \{\boldsymbol{\Delta'} : \boldsymbol{0} \leq \boldsymbol{\Delta'} \leq \boldsymbol{\delta}(\boldsymbol{p})^E\}$.

Time $t \in [0, \infty)$ is called a *step* of the price adjustment procedure if there exist $i \in N$, $\boldsymbol{x_i} \in Q_i(\boldsymbol{p}(t))$, and $t' \in [0, t)$ such that for each $t'' \in (t', t)$

$$\boldsymbol{x_i} \notin Q_i(\boldsymbol{p}(t'')).$$

Lemma 3. *Suppose that assumptions $A1-A4$ hold, and that bidders truthfully report their demand. Then, for each $\boldsymbol{p} \in \mathbb{R}^K_+$,*

$$E_+(\boldsymbol{p}) \in argmin_{E\subset K} \sum_{k\in E}(S_k - \sum_{i\in N} \widehat{x}_i^k(\boldsymbol{p}, E)), \tag{27}$$

and there exists $\overline{\delta} > 0$ such that for each $\delta \in [0, \overline{\delta})$,

$$E_+(\boldsymbol{p} + \boldsymbol{\delta}^{E_+(\boldsymbol{p})}) = E_+(\boldsymbol{p}).$$

Similarly, for each $\boldsymbol{p} \in \mathbb{R}^{\mathbb{K}}_+$,

$$E_-(\boldsymbol{p}) \in argmax_{E\subset K} \sum_{k\in E}(S_k - \sum_{i\in N} \widehat{y}_i^k(\boldsymbol{p}, E)),$$

and there exists $\overline{\delta}' > 0$ such that for each $\delta \in [0, \overline{\delta}')$,

$$E_-(\boldsymbol{p} - \boldsymbol{\delta}^{E_-(\boldsymbol{p})}) = E_-(\boldsymbol{p}).$$

Moreover, if t and t' are two consecutive steps of the EAPAP such that $t < t'$, then for all $\hat{t} \in [t, t')$

$$E_+(\boldsymbol{p}(\hat{t})) = E_+(\boldsymbol{p}(t)),$$

and if t and t' are two consecutive steps of the EDPAP such that $t < t'$, then for all $\hat{t} \in [t, t')$

$$E_-(\boldsymbol{p}(\hat{t})) = E_-(\boldsymbol{p}(t)).$$

Proof. For each $\boldsymbol{p} \in \mathbb{R}^K_+$, by Propositions 2 and 3, there exist $\delta(\boldsymbol{p}) \in \mathbb{R}_{++}$ and $E_+(\boldsymbol{p}) \subset K$ such that $\boldsymbol{p}_+ = \boldsymbol{p} + \boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})}$. Note that

$$\boldsymbol{\delta}(\boldsymbol{p})^{E_+(\boldsymbol{p})} \cdot \boldsymbol{x} = \delta(\boldsymbol{p}) \sum_{k\in E_+(\boldsymbol{p})} x^k.$$

Therefore, by Proposition 2 and by Definition of $\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, E_+(\boldsymbol{p}))$ in equation 25, for each $i \in N$, and for each $\delta \in (0, \delta(\boldsymbol{p})]$

$$\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p}, E_+(\boldsymbol{p})) \in Q_i(\boldsymbol{p} + \boldsymbol{\delta}^{E_+(\boldsymbol{p})}).$$

Observe that

$$\begin{aligned}
L(\boldsymbol{p}+\boldsymbol{\delta}^{E_+(\boldsymbol{p})}) =&(\boldsymbol{p}+\boldsymbol{\delta}^{E_+(\boldsymbol{p})})\cdot\boldsymbol{S}\\
&+\sum_{i\in N}(U_i(\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p})))-(\boldsymbol{p}+\boldsymbol{\delta}^{E_+(\boldsymbol{p})})\cdot\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p}))))\\
=&\boldsymbol{p}\cdot\boldsymbol{S}+\sum_{i\in N}(U_i(\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p}))-\boldsymbol{p}\cdot\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p})))+\boldsymbol{\delta}^{E_+(\boldsymbol{p})}\cdot\boldsymbol{S}\\
&-\sum_{i\in N}\boldsymbol{\delta}^{E_+(\boldsymbol{p})}\cdot\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p})))\\
=&L(\boldsymbol{p})+\boldsymbol{\delta}^{E_+(\boldsymbol{p})}\cdot\boldsymbol{S}-\sum_{i\in N}\boldsymbol{\delta}^{E_+(\boldsymbol{p})}\cdot\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E_+(\boldsymbol{p}))).
\end{aligned}$$

So, minimizing $L(\boldsymbol{p}+\boldsymbol{\delta}^E)$, and maximizing $L(\boldsymbol{p})-L(\boldsymbol{p}+\boldsymbol{\delta}^E))$ over the set of goods $E\subset K$ is equivalent to

$$\min_{E\subset K}\boldsymbol{\delta}^E\cdot\boldsymbol{S}-\sum_{i\in N}\boldsymbol{\delta}^E\cdot\widehat{\boldsymbol{x}}_{\boldsymbol{i}}(\boldsymbol{p},E))=\delta\sum_{k\in E}(S_k-\sum_{i\in N}\widehat{x}_i^k(\boldsymbol{p},E)).$$

That is why

$$E_+(\boldsymbol{p})\in argmin_{E\subset K}\sum_{k\in E}(S_k-\sum_{i\in N}\widehat{x}_i^k(\boldsymbol{p},E)).$$

Therefore, the following holds:

$$\widehat{\boldsymbol{x}}\in\arg\min_{\boldsymbol{x_i}\in Q_i(\boldsymbol{p})}\sum_{k\in E}x_i^k \text{ if and only if } \widehat{\boldsymbol{x}}\in\arg\min_{\boldsymbol{x_i}\in Q_i(\boldsymbol{p}+\boldsymbol{\delta}^{E_+(\boldsymbol{p})})}\sum_{k\in E}x_i^k.$$

Hence, $E_+(\boldsymbol{p}+\boldsymbol{\delta}^{E_+(\boldsymbol{p})})=E_+(\boldsymbol{p})$. So, for each step t, there exist $\overline{\delta}>0$ and $E_+(\boldsymbol{p}(t))\subset K$ such that for each $\delta\in(0,\overline{\delta})$

$$E_+(\boldsymbol{p}(t)+\boldsymbol{\delta}^{E_+(\boldsymbol{p}(t))})=E_+(\boldsymbol{p}(t)).$$

Let t' be a step such that $t'>t$, and t and t' are consecutive steps. As there is no step between $\boldsymbol{p}(t)$ and $\boldsymbol{p}(t)+\overline{\boldsymbol{\delta}}^{E_+(\boldsymbol{p}(t))}$, and because for each $\hat{t}\in(t,t')$ and for each $i\in N$, $Q_i(\boldsymbol{p}(\hat{t}))=Q_i(\boldsymbol{p}(t))$ holds,

$$E_+(\boldsymbol{p}(\hat{t}))=E_+(\boldsymbol{p}(t)).$$

An analogous derivation can be made for $E_-(\boldsymbol{p})$. Observe that at equilibrium price vector $\boldsymbol{p^*}$, $E_+(\boldsymbol{p^*})=E_-(\boldsymbol{p^*})=\emptyset$. $\square$

Note that Lemma 3 implies that in the EAPAP and the EDPAP, bidders reporting their demand sets at the initial price vector and at prices at which they add a bundle to their demand sets gives sufficient information to the auctioneer to adjust prices correctly.

4.2 Reaching Walrasian Equilibrium Prices in Finite Steps

The following Propositions show properties of the EAPAP and the EDPAP, and their convergence. Theorems 2 and 3 show that the EAPAP and the EDPAP converge to Walrasian equilibrium prices.

Proposition 4. *Suppose that assumptions A1-A4 hold, and that bidders truthfully report their demand. Starting from any initial price vector* $\boldsymbol{p}(0) \in \mathbb{R}^K_+$*, if the EAPAP ends at* T*, then* $\boldsymbol{p}(T) \geq \underline{\boldsymbol{p}}$*. Similarly, starting from any initial price* $\boldsymbol{p}(0) \in \mathbb{R}^K_+$*, if the EDPAP ends at* T*, then* $\boldsymbol{p}(T) \leq \overline{\boldsymbol{p}}$*.*

Proof. Suppose, for the EAPAP, that $\boldsymbol{p}(T) \ngeq \underline{\boldsymbol{p}}$. Then, $\boldsymbol{p}(T) \wedge \underline{\boldsymbol{p}}$ is less than $\underline{\boldsymbol{p}}$ in at least one component. Since $\underline{\boldsymbol{p}}$ is the smallest Walrasian equilibrium price vector, $\boldsymbol{p}(T) \wedge \underline{\boldsymbol{p}}$ is not a Walrasian equilibrium price vector of the economy. Therefore, by Proposition 1

$$L(\underline{\boldsymbol{p}}) < L(\boldsymbol{p}(T) \wedge \underline{\boldsymbol{p}}). \tag{28}$$

As $L(\cdot)$ is submodular by Lemma 2,

$$L(\boldsymbol{p}(T) \wedge \underline{\boldsymbol{p}}) + L(\boldsymbol{p}(T) \vee \underline{\boldsymbol{p}}) \leq L(\boldsymbol{p}(T)) + L(\underline{\boldsymbol{p}}). \tag{29}$$

Inequalities 28 and 29 imply that

$$L(\boldsymbol{p}(T) \vee \underline{\boldsymbol{p}}) < L(\boldsymbol{p}(T)). \tag{30}$$

By Propositions 2 and 3, and by the definition of $\delta(\cdot)$ in equation 17, there exist $\delta(\boldsymbol{p}(T)) \in \mathbb{R}_{++}$ and $E(\boldsymbol{p}(T)) \subset K$ such that $\boldsymbol{p}(T) + \boldsymbol{\delta}(\boldsymbol{p})^{E(\boldsymbol{p}(T))}$ is the unique minimal minimizer of $L(\cdot)$ in the cube $\{\boldsymbol{p} + \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p}(T))^K\}$. Let $\boldsymbol{p'}$ be a strict convex combination of $\boldsymbol{p}(T)$ and $\boldsymbol{p}(T) \vee \underline{\boldsymbol{p}}$ such that $\boldsymbol{0} \leq \boldsymbol{p'} - \boldsymbol{p}(T) \leq \boldsymbol{\delta}(\boldsymbol{p}(T))^K$. $\boldsymbol{p'}$ exists as $\boldsymbol{p}(T) \vee \underline{\boldsymbol{p}} \gneq \boldsymbol{p}(T)$. Convexity of $L(\cdot)$ implies that $L(\boldsymbol{p'}) < L(\boldsymbol{p}(T))$. This means that $E(\boldsymbol{p}(T)) \neq \emptyset$ and that $\boldsymbol{p}(T)$ is not a minimizer in $\{\boldsymbol{p}(T) + \boldsymbol{\Delta} : \boldsymbol{0} \leq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p}(T))^K\}$. Therefore, the EAPAP does not stop at T as it is possible to decrease the Lyapunov function further, a contradiction. Hence, $\boldsymbol{p}(T) \geq \underline{\boldsymbol{p}}$.

The result for the EDPAP can be proven analogously. □

Proposition 5. *Suppose that assumptions* $A1 - A4$ *hold, and that bidders truthfully report their demand. In the EAPAP, starting from any initial price vector* $\boldsymbol{p}(0) \in \mathbb{R}_+^K$*, if* $\boldsymbol{p}(t) \leq \underline{\boldsymbol{p}}$*, then* $\boldsymbol{p}(t') \leq \underline{\boldsymbol{p}}$ *for all* $t' > t$*. In the EDPAP, starting from any initial price vector* $\boldsymbol{p}(0) \in \mathbb{R}_+^K$*, if* $\boldsymbol{p}(t) \geq \overline{\boldsymbol{p}}$*, then* $\boldsymbol{p}(t') \geq \overline{\boldsymbol{p}}$ *for all* $t' > t$*.*

Proof. Consider the EAPAP. Suppose, on the contrary, that there exist t and t' such that $t' > t$, $\boldsymbol{p}(t) \leq \underline{\boldsymbol{p}}$, and $p(t')^k > \underline{p}^k$ for some $k \in K$. As $\boldsymbol{p}(\cdot)$ is continuous, there exists t'' such that $t < t'' < t'$ and $p(t'')^k = \underline{p}^k$. Let

$$\overline{t''} = \sup_{\boldsymbol{p}(t) \leq \underline{\boldsymbol{p}}} t. \tag{31}$$

Since $\boldsymbol{p}(\cdot)$ is continuous, and prices are ascending, $\overline{t''}$ exists and $p(\overline{t''})^k = \underline{p}^k$. By Proposition 1, and as $\underline{\boldsymbol{p}}$ is a Walrasian price vector, for each $s > \overline{t''}$,

$$L(\underline{\boldsymbol{p}}) \leq L(\boldsymbol{p}(s) \vee \underline{\boldsymbol{p}}). \tag{32}$$

Moreover, since $L(\cdot)$ is a submodular function by Lemma 2,

$$L(\boldsymbol{p}(s) \vee \underline{\boldsymbol{p}}) + L(\boldsymbol{p}(s) \wedge \underline{\boldsymbol{p}}) \leq L(\boldsymbol{p}(s)) + L(\underline{\boldsymbol{p}}). \tag{33}$$

Inequalities 32 and 33 imply that

$$L(\boldsymbol{p}(s) \wedge \underline{\boldsymbol{p}}) \leq L(\boldsymbol{p}(s)). \tag{34}$$

Note that there exists $k' \in K$ such that for each $s > \overline{t''}$

$$p^{k'}(s) \wedge \underline{p}^{k'} < p^{k'}(s). \tag{35}$$

By Proposition 3, there exists $E \subset K$ such that

$$\boldsymbol{p}(\overline{s}) = \boldsymbol{p}(\overline{t''}) + \boldsymbol{\delta}(\boldsymbol{p}(\overline{t''}))^E. \tag{36}$$

for some $\overline{s} \in (\overline{t''}, t')$. Since inequality 35 holds for $s = \overline{s}$,

$$\boldsymbol{p}(\overline{s}) \wedge \underline{\boldsymbol{p}} = \boldsymbol{p}(\overline{t''}) + \boldsymbol{\delta}(\boldsymbol{p}(\overline{t''}))^A$$

for some $A \subsetneq E$. Therefore, by inequality 34, $\boldsymbol{p}(\overline{s})$ is not a minimal minimizer, a contradiction.

The result for the EDPAP can be proven analogously. □

Next Lemma shows that as a set of prices are increased, if a bidder removes a bundle from his demand set, he does so immediately. An analogous result is also true for the EDPAP.

Lemma 4. *Let* $\boldsymbol{x_i}, \boldsymbol{x'_i} \in X_i$ *be such that* $U_i(\boldsymbol{x_i}) - \boldsymbol{p} \cdot \boldsymbol{x_i} = U_i(\boldsymbol{x'_i}) - \boldsymbol{p} \cdot \boldsymbol{x'_i}$ *for some* $i \in N$*, and some* $\boldsymbol{p} \in \mathbb{R}^K_+$*. If there exists* $\delta > 0$ *such that*

$$U_i(\boldsymbol{x_i}) - (\boldsymbol{p} + \boldsymbol{\delta}^E) \cdot \boldsymbol{x_i} > U_i(\boldsymbol{x'_i}) - (\boldsymbol{p} + \boldsymbol{\delta}^E) \cdot \boldsymbol{x'_i},$$

then for each $\delta' > 0$

$$U_i(\boldsymbol{x_i}) - (\boldsymbol{p} + \boldsymbol{\delta'}^E) \cdot \boldsymbol{x_i} > U_i(\boldsymbol{x'_i}) - (\boldsymbol{p} + \boldsymbol{\delta'}^E) \cdot \boldsymbol{x'_i}.$$

Proof. Suppose that $\boldsymbol{p} \in \mathbb{R}^K_+$, $\boldsymbol{x_i}, \boldsymbol{x'_i} \in X_i$ such that

$$U_i(\boldsymbol{x_i}) - \boldsymbol{p} \cdot \boldsymbol{x_i} = U_i(\boldsymbol{x'_i}) - \boldsymbol{p} \cdot \boldsymbol{x'_i}. \tag{37}$$

Then, by Lemma 37, for each $\delta > 0$

$$U_i(\boldsymbol{x_i}) - (\boldsymbol{p} + \boldsymbol{\delta}^E) \cdot \boldsymbol{x_i} > U_i(\boldsymbol{x'_i}) - (\boldsymbol{p} + \boldsymbol{\delta}^E) \cdot \boldsymbol{x'_i}$$

if and only if

$$0 > \delta \sum_{k \in E} (\boldsymbol{x_i^k} - \boldsymbol{x_i'^k}).$$

If these inequalities hold for some $\delta > 0$, then

$$0 > \delta' \sum_{k \in E} (\boldsymbol{x_i^k} - \boldsymbol{x_i'^k}) \text{ for all } \delta' > 0.$$

Hence, the result follows. ☐

Theorem 1 shows that the EAPAP and the EDPAP converge in finite steps. Moreover, changes in prices in between steps are identified.

Theorem 1. *Suppose that assumptions* A1−A4 *hold, and that bidders truthfully report their demands. Then, the EAPAP and the EDPAP terminate in finite steps. The increase* δ_s *in prices in the EAPAP at step* s *is*

$$\delta_s = U_{i_s}(\boldsymbol{x_{i_s}}) - U_{i_s}(\boldsymbol{x'_{i_s}}) - p_s^k, \tag{38}$$

or

$$\delta_s = U_{i_s}(\boldsymbol{x_{i_s}}) - U_{i_s}(\boldsymbol{x'_{i_s}}) - p_s^k + p_s^{k'}. \tag{39}$$

for some bidder i_s*, some bundles* $\boldsymbol{x'_{i_s}}$*,* $\boldsymbol{x_{i_s}}$*, and some goods* k*,* k' *such that* $\#(\boldsymbol{x_{i_s}}, \boldsymbol{x'_{i_s}}) \leq 1$*,* p_s^k *and* $p_s^{k'}$ *are the prices of goods* k *and* k'*, respectively, at step* s*, and* i_s *is a bidder who demands* $\boldsymbol{x_{i_s}}$ *at* $\boldsymbol{p_s}$*, and adds a bundle,* $\boldsymbol{x'_{i_s}}$*, to his demand set at price vector* $\boldsymbol{p_{s+1}}$*.*

Proof. Consider the EAPAP. As there are finite number of bidders and the consumption set of each bidder is bounded, there is an upper bound on bidders' values for each good. In other words, for each good $k \in K$, there is a price $p^k_{max} \in \mathbb{R}_+$ at and above which no bidder wants good k regardless of the prices of the rest of the goods . Observe that, by Lemma 3, if the total quantity demanded of a good $k \in K$ at price vector $\boldsymbol{p}(t) \in \mathbb{R}^K_+$ is less than the total quantity of good k available, then the price of good k will not be in the set of prices that will be increased at t in the EAPAP. Therefore, in the EAPAP, for each $k \in K$ and for each $t \in [0, \infty)$,

$$p^k(t) \leq p^k_{max}.$$

Now, I will show that the EAPAP converges in finite steps. By Corollary 2, for all consecutive steps t and t' such that $t' < t$, there exists a rational number $t_q \in \mathbb{Q}$ such that $t' < t_q < t$. Therefore, there are at most countably many steps in the EAPAP. Let $\{\boldsymbol{p}_s\}_{s \in \sigma}$ be the sequence of all price vectors reached by the EAPAP at all steps.

Observe that, by construction of the EAPAP, if $t' > t''$, then $\boldsymbol{p}(t') \gneq \boldsymbol{p}(t'')$. Therefore, inequality **??** is always strict, i.e., for each $s \in \sigma$,

$$\boldsymbol{p_s} \lneq \boldsymbol{p_{s+1}}. \tag{40}$$

As the sequence $\{\boldsymbol{p_s}\}_{s \in \sigma}$ is bounded above and monotonically increasing, it converges to a price vector $\boldsymbol{p^*} \in \mathbb{R}_+$ such that $\boldsymbol{p^*} \leq (p^k_{max})_{k \in K}$.

Note that for each $s \in \sigma$, $\boldsymbol{p_{s+1}}$ is the first price vector where a bidder adds a bundle to his demand set after $\boldsymbol{p_s}$ in the EAPAP. Since the set of prices that are increased does not change between consecutive steps in the EAPAP, the same set of prices are increased from $\boldsymbol{p_s}$ to $\boldsymbol{p_{s+1}}$. Therefore, there exist $\delta_s > 0$ and $E_s \subset K$ such that

$$\boldsymbol{p_{s+1}} = \boldsymbol{p_s} + \boldsymbol{\delta_s}^{E_s}, \tag{41}$$

which can be rewritten as

$$\boldsymbol{p_{s+1}} = \boldsymbol{p_0} + \sum_{s'=0}^{s} \boldsymbol{\delta_{s'}}^{E_{s'}} \tag{42}$$

where $\delta_{s'} > 0$ and $E_{s'} \subset K$ for each $s' \in \mathbb{Z}$ such that $0 \leq s' \leq s$.

I will now show that for each step $s \in \sigma$, and for each bidder $i_s \in N$ who adds a bundle, $\boldsymbol{x'_{i_s}} \in X_i$, to his demand set at price vector $\boldsymbol{p_{s+1}}$, there exists a bundle $\boldsymbol{x_{i_s}}$ such that $\boldsymbol{x_{i_s}}$ is in i_s's demand set at all prices reached between $\boldsymbol{p_s}$ and $\boldsymbol{p_{s+1}}$, and $\boldsymbol{x_{i_s}}$ can be formed by adding at most one unit of a good to and removing at most one unit of another good from $\boldsymbol{x'_{i_s}}$.

As the same set of prices increase from $\boldsymbol{p_s}$ to $\boldsymbol{p_{s+1}}$, and since X_{i_s} is finite, there exist $t' \in [0, \infty)$ such that $\boldsymbol{p_s} \leq \boldsymbol{p}(t') < \boldsymbol{p_{s+1}}$, and for each $\boldsymbol{y} \in X_{i_s}$ and for each $\widetilde{t} \geq t'$, if

$$\boldsymbol{p}(t') \leq \boldsymbol{p}(\widetilde{t}) < \boldsymbol{p_{s+1}} \text{ and } U_{i_s}(\boldsymbol{y}) - \boldsymbol{p}(\widetilde{t}) \cdot \boldsymbol{y} > U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p}(\widetilde{t}) \cdot \boldsymbol{x'_{i_s}}, \tag{43}$$

then

$$\boldsymbol{y} \in Q_{i_s}(\boldsymbol{p}(\widetilde{t})). \tag{44}$$

Therefore, by the *single-improvement* property (implied by Assumption *A*4, the *gross substitutes*, see Lemma 2 in Gul and Stacchetti [6]), there exists $\boldsymbol{x_{i_s}} \in X_{i_s}$ such that

$$\#(\boldsymbol{x_{i_s}} \setminus \boldsymbol{x'_{i_s}}) \leq 1 \text{ and } \#(\boldsymbol{x'_{i_s}} \setminus \boldsymbol{x_{i_s}}) \leq 1, \tag{45}$$

and for each $\widetilde{t} \geq t'$ such that $\boldsymbol{p_s} \leq \boldsymbol{p}(\widetilde{t}) < \boldsymbol{p_{s+1}}$

$$\boldsymbol{x_{i_s}} \in Q_{i_s}(\boldsymbol{p}(\widetilde{t})). \tag{46}$$

As the same set of prices increase from $\boldsymbol{p_s}$ to $\boldsymbol{p_{s+1}}$, for each $\widetilde{t} \in [0, \infty)$ such that $\boldsymbol{p_s} \leq \boldsymbol{p}(\widetilde{t}) \leq \boldsymbol{p_{s+1}}$

$$\boldsymbol{x_{i_s}} \in Q_{i_s}(\boldsymbol{p}(\widetilde{t})) \tag{47}$$

and

$$\boldsymbol{x'_{i_s}} \notin Q_{i_s}(\boldsymbol{p}(\widetilde{t})). \tag{48}$$

Hence,

$$\boldsymbol{x_{i_s}} \in Q(\boldsymbol{p_s}),\ \boldsymbol{x'_{i_s}} \notin Q(\boldsymbol{p_s}) \text{ and } \boldsymbol{x_{i_s}}, \boldsymbol{x'_{i_s}} \in Q_{i_s}(\boldsymbol{p_{s+1}}). \tag{49}$$

Now I claim that equation 41, inequalities 45, and equations 49 imply that there exist a unique $k \in E_s$ such that

$$x^k_{i_s} \neq x'^k_{i_s}. \tag{50}$$

Suppose, on the contrary, that the claim is false. Then, there would be two possibilities by inequalities 45. In the first one, for each $\widetilde{k} \in E_s$

$$x^{\widetilde{k}}_{i_s} = x'^{\widetilde{k}}_{i_s}, \tag{51}$$

implying that

$$\boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x_{i_s}} = \boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x'_{i_s}}. \tag{52}$$

Therefore,

$$\begin{aligned} U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x_{i_s}} - (U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x'_{i_s}}) =& U_{i_s}(\boldsymbol{x_{i_s}}) - (\boldsymbol{p_s} + \boldsymbol{\delta_s}^{E_s}) \cdot \boldsymbol{x_{i_s}} \\ & - (U_{i_s}(\boldsymbol{x'_{i_s}}) - (\boldsymbol{p_s} + \boldsymbol{\delta_s}^{E_s}) \cdot \boldsymbol{x'_{i_s}}), \end{aligned} \tag{53}$$

and

$$\begin{aligned} U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x_{i_s}} - (U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x'_{i_s}}) =& U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_{s+1}} \cdot \boldsymbol{x_{i_s}} \\ & - (U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_{s+1}} \cdot \boldsymbol{x'_{i_s}}), \end{aligned} \tag{54}$$

a contradiction to equations 49.

In the second one, there exist $\widetilde{k}, \widetilde{k'} \in E_s$ such that $\widetilde{k} \neq \widetilde{k'}$, $x_{i_s}^{\widetilde{k}} \neq x_{i_s}'^{\widetilde{k}}$ and $x_{i_s}^{\widetilde{k'}} \neq x_{i_s}'^{\widetilde{k'}}$ hold, which, by inequalities 45, also imply equations 52, 53 and 54, contradicting equations 49. Hence, the claim is true. Observe that since there exists a unique $k \in E_s$ satisfying inequality 50, inequalities 45 and equations 49 imply that

$$x_{i_s}^{k} = x_{i_s}'^{k} + 1. \tag{55}$$

By inequalities 45, there exists at most one $k' \in K \setminus E_s$ such that

$$x_{i_s}^{k'} + 1 = x_{i_s}'^{k'}.$$

For each $s \in \sigma$, let $t_s \in [0, \infty)$ be such that

$$\boldsymbol{p}_s = \boldsymbol{p}(t_s).$$

Now I will show that the total number of goods in excess demand strictly decreases at each step. As the prices in E_s are increased from $\boldsymbol{p_s}$ to $\boldsymbol{p_{s+1}}$, by Lemma 4, for each $i \in N$ and for all $t, t' \in (t_s, t_{s+1})$

$$Q_i(\boldsymbol{p}(t)) = Q_i(\boldsymbol{p}(t')),$$

and for each $t \in [t_s, t_{s+1})$

$$Q_i(\boldsymbol{p_s}) \supset Q_i(\boldsymbol{p}(t))$$

and by definition of step,

$$Q_i(\boldsymbol{p}(t)) \subset Q_i(\boldsymbol{p_{s+1}}). \tag{56}$$

Therefore, for each $t \in (t_s, t_{s+1})$

$$Q_i(\boldsymbol{p}(t)) \subset Q_i(\boldsymbol{p_s}) \cap Q_i(\boldsymbol{p_{s+1}}),$$

and by Lemma 4

$$Q_i(\boldsymbol{p}(t)) = Q_i(\boldsymbol{p_s}) \cap Q_i(\boldsymbol{p_{s+1}}).$$

For each $i \in N$ and for each $t \in (t_s, t_{s+1})$, by equation 56, and by definition of $\widehat{\boldsymbol{x}}_i(\cdot)$ in equation 25,

$$\sum_{\hat{k} \in E_{s+1}} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_{s+1}}, E_{s+1}) \leq \sum_{\hat{k} \in E_{s+1}} \widehat{x}_i^{\hat{k}}(\boldsymbol{p}(t), E_{s+1}). \tag{57}$$

Therefore, for each $t \in (t_s, t_{s+1})$

$$\sum_{\hat{k} \in E_{s+1}} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_{s+1}}, E_{s+1})) \geq \sum_{\hat{k} \in E_{s+1}} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p}(t), E_{s+1})).$$

Since, by Lemma 3, for each $t \in (t_s, t_{s+1})$

$$\sum_{\hat{k} \in E_{s+1}} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p}(t), E_{s+1})) \geq \sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p}(t), E_s))$$

and

$$\sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p}(t), E_s)) = \sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_s}, E_s),$$

the following inequality holds

$$\sum_{\hat{k} \in E_{s+1}} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_{s+1}}, E_{s+1})) \geq \sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_s}, E_s)). \tag{58}$$

Suppose that for each $i \in N$ there exists $\tilde{\boldsymbol{x}}_{\boldsymbol{i}} \in Q_i(\boldsymbol{p_s}) \cap Q_i(\boldsymbol{p_{s+1}})$ satisfying equation 25 at $(\boldsymbol{p_{s+1}}, E_{s+1})$, then

$$\sum_{\hat{k} \in E_{s+1}} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_{s+1}}, E_{s+1})) = \sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \tilde{x}_i^{\hat{k}}).$$

As

$$\sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_s}, E_s)) = \sum_{\hat{k} \in E_s} (S_{\hat{k}} - \sum_{i \in N} \tilde{x}_i^{\hat{k}}),$$

by inequality 58, $E_s = E_{s+1}$. But equation 55 implies that for each bidder i_s who adds a bundle at step t_{s+1}

$$\sum_{\hat{k}\in E_s} \tilde{x}_{i_s}^{\hat{k}} > \sum_{\hat{k}\in E_s} x_{i_s}^{\prime\hat{k}} \tag{59}$$

because

$$\sum_{\hat{k}\in E_s} x_{i_s}^{\hat{k}} > \sum_{\hat{k}\in E_s} x_{i_s}^{\prime\hat{k}},$$

and

$$\sum_{\hat{k}\in E_s} \tilde{x}_{i_s}^{\hat{k}} = \sum_{\hat{k}\in E_s} x_{i_s}^{\hat{k}},$$

where $\boldsymbol{x'_{i_s}}$ is any one of the bundles bidder i_s adds to his demand set at $\boldsymbol{p_{s+1}}$, and $\boldsymbol{x_{i_s}} \in Q_{i_s}(\boldsymbol{p_s}) \cap Q_{i_s}(\boldsymbol{p_{s+1}})$ such that $\boldsymbol{x'_{i_s}}$ and $\boldsymbol{x_{i_s}}$ satisfy equations 45. If $E_s = E_{s+1}$, then inequality 59 implies that $\boldsymbol{\tilde{x}_{i_s}} \in Q_{i_s}(\boldsymbol{p_s}) \cap Q_{i_s}(\boldsymbol{p_{s+1}})$ does not satisfy equation 25 at $(\boldsymbol{p_{s+1}}, E_{s+1})$, a contradiction.

Therefore, there exists a bidder $i'_s \in N$ such that there does not exist $\boldsymbol{\tilde{x}_{i'_s}} \in Q_{i'_s}(\boldsymbol{p_s}) \cap Q_{i'_s}(\boldsymbol{p_{s+1}})$ satisfying equation 25 at $(\boldsymbol{p_{s+1}}, E_{s+1})$. So, bidder i'_s adds a bundle to his demand set at $\boldsymbol{p_{s+1}}$, and there exists $\boldsymbol{x'_{i'_s}} \in Q_{i'_s}(\boldsymbol{p_{s+1}})$ satisfying equation 25 at $(\boldsymbol{p_{s+1}}, E_{s+1})$.

Hence,

$$\sum_{\hat{k}\in E_{s+1}} x_{i'_s}^{\prime\hat{k}} < \sum_{\hat{k}\in E_{s+1}} \tilde{x}_{i'_s}^{\hat{k}} \tag{60}$$

for all $\boldsymbol{\tilde{x}_{i'_s}} \in Q_{i'}(\boldsymbol{p_s}) \cap Q_{i'}(\boldsymbol{p_{s+1}})$.

Inequality 60 implies that inequality 57 is strict for i'_s. So, inequality 58 is strict. So, for each $t \in (t_s, t_{s+1})$,

$$\sum_{\hat{k}\in E_{s+1}} (S_{\hat{k}} - \sum_{i\in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_{s+1}}, E_{s+1})) > \sum_{\hat{k}\in E_s} (S_{\hat{k}} - \sum_{i\in N} \widehat{x}_i^{\hat{k}}(\boldsymbol{p_s}, E_s)). \tag{61}$$

As the left side of inequality 61 is integer and since the EAPAP terminates whenever it is positive, there are finitely many steps.

Now I will show that the price of each good at each step can be written as an integer combination of bidders' differences in utilities between bundles satisfying inequalities 45, and initial prices.

Therefore,

$$\boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x_{i_s}} - \boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x'_{i_s}} = \delta_s. \tag{62}$$

Equations 49 imply that

$$U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_{s+1}} \cdot \boldsymbol{x_{i_s}} = U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_{s+1}} \cdot \boldsymbol{x'_{i_{s+1}}}. \tag{63}$$

By equation 41, equation 63 implies that

$$U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x_{i_s}} - \boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x_{i_s}} = U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x'_{i_{s+1}}} - \boldsymbol{\delta_s}^{E_s} \cdot \boldsymbol{x'_{i_s}}, \tag{64}$$

and by equation 62

$$\begin{aligned} \delta_s &= U_{i_s}(\boldsymbol{x_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x_{i_s}} - (U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_s} \cdot \boldsymbol{x'_{i_s}}) \\ &= U_{i_s}(\boldsymbol{x_{i_s}}) - U_{i_s}(\boldsymbol{x'_{i_s}}) - \boldsymbol{p_s} \cdot (\boldsymbol{x_{i_s}} - \boldsymbol{x'_{i_s}}). \end{aligned} \tag{65}$$

By inequalities 45, it is either

$$\delta_s = U_{i_s}(\boldsymbol{x_{i_s}}) - U_{i_s}(\boldsymbol{x'_{i_s}}) - p^k_s, \tag{66}$$

or

$$\delta_s = U_{i_s}(\boldsymbol{x_{i_s}}) - U_{i_s}(\boldsymbol{x'_{i_s}}) - p^k_s + p^{k'}_s. \tag{67}$$

Therefore, for each $\hat{s} \in \sigma$ and for each $\hat{k} \in K$, $p^{\hat{k}}_{\hat{s}}$ can be written as integer combinations of bidders' differences in utilities between bundles satisfying inequalities 45 and initial prices, i.e., for each $\hat{s} \in \sigma$ and for each $\hat{k} \in K$

$$p^{\hat{k}}(t) = \sum_{i \in N} \sum_{\substack{(\boldsymbol{x_i}, \boldsymbol{x'_i}) \in X_i \times X_i \\ \text{s.t. } \#(\boldsymbol{x_i}, \boldsymbol{x'_i}) \leq 1 \\ \text{and } \#(\boldsymbol{x'_i}, \boldsymbol{x_i}) \leq 1}} \left[a^{(\hat{k},t)}_{(i,\boldsymbol{x_i},\boldsymbol{x'_i})} \left[U_i(\boldsymbol{x_i}) - U_i(\boldsymbol{x'_i}) \right] \right] - \sum_{k \in K} b^{(\hat{k},t)} p^k_0 \tag{68}$$

where all coefficients $a^{(\hat{k},t)}_{(i,\boldsymbol{x_i},\boldsymbol{x'_i})}, b^{(\hat{k},t)} \in \mathbb{Z}$, and $t \in [0, \infty)$ is the step corresponding to $\hat{s} \in \sigma$.

Observe that the definition of *step* implies that if there is no market clearing allocation at a *step*, then there will not be a market clearing allocation until the next step. Therefore, the EAPAP terminates at price vector $\boldsymbol{p_{|\sigma|-1}} = \boldsymbol{p}(T) = \boldsymbol{p^*} \in \mathbb{R}^K_+$ for some finite T.

The proof for the EDPAP can be done analogously. $\square$

Theorem 2 states the sufficient conditions for the EAPAP to reach a Walrasian equilibrium price vector. Analogously, Theorem 3 states the sufficient conditions for the EDPAP to reach a Walrasian equilibrium price vector.

Theorem 2. *Suppose that assumptions $A1 - A4$ hold, and that bidders truthfully report their demands. Then, starting from any initial price vector of $\boldsymbol{p}(0) \in \mathbb{R}_+^K$ such that $\boldsymbol{p}(0) \leq \underline{\boldsymbol{p}}$, the EAPAP reaches the lowest Walrasian equilibrium price vector $\underline{\boldsymbol{p}}$ in finite* steps*.*

Proof. The auctioneer asks each bidder $i \in N$ his demand set $x_i(\boldsymbol{p}(0)) \subset X_i$ at $\boldsymbol{p}(0)$. Using these demand sets, the auctioneer determines the set $E_+(\boldsymbol{p}(0)) \subset K$ of goods in excess demand at $\boldsymbol{p}(0)$ (see Lemma 3). Prices of these goods in $E_+(\boldsymbol{p}(0))$ are increased continuously at the same rate while the rest remains the same. As prices are increased, at any time $t \in [0, \infty)$, if there is a bidder who adds a bundle to his demand set, then the price adjustment stops. Each bidder i reports his demand $x_i(\boldsymbol{p}(t)) \subset X_i$ at $\boldsymbol{p}(t)$. The auctioneer determines the set $E_+(\boldsymbol{p}(t)) \subset K$ of goods in excess demand at $\boldsymbol{p}(t)$ (see Lemma 3). Prices of goods in $E_+(\boldsymbol{p}(t))$ are increased continuously at the same rate while the rest remains the same. The EAPAP reaches some price vector in finite steps (see Theorem 1). By Propositions 4 and 5, the price vector the EAPAP reaches is $\underline{\boldsymbol{p}}$, and by Proposition 1, there exists a market clearing allocation $(\boldsymbol{x_i^*})_{i \in N}$ such that $\boldsymbol{x_i^*} \in x_i(\underline{\boldsymbol{p}})$ for each $i \in N$, provided that bidders report their demands truthfully. $\square$

The proof of the convergence of the EDPAP to the highest Walrasian equilibrium price, Theorem 3 below, is analogous to the proof of Theorem 2.

Theorem 3. *Suppose that assumptions $A1 - A4$ hold, and that bidders truthfully report their demands. Then, starting from any initial price vector of $\boldsymbol{p}(0) \in \mathbb{R}_+^K$ such that $\boldsymbol{p}(0) \geq \overline{\boldsymbol{p}}$, the EDPAP reaches the highest Walrasian equilibrium price vector $\overline{\boldsymbol{p}}$ in finite* steps*.*

Previous Theorems impose restrictions on initial price vectors. If these conditions are violated, then the EAPAP and the EDPAP converge in finite steps, by Theorem 1, to *supporting price vectors* by Theorem 4 whenever bidders report their demand sets truthfully.

A price vector $\boldsymbol{p'} \in \mathbb{R}_+^K$ is a *supporting price vector* if there exists an allocation $\boldsymbol{x'} = (\boldsymbol{x'_i})_{i \in N}$ such that $\sum_{i \in N} \boldsymbol{x_i} \leq \boldsymbol{S}$, and for each $i \in N$, $\boldsymbol{x'_i} \in Q_i(\boldsymbol{p'})$. The next Theorem shows that the EDPAP converges to a *supporting price vector.*

Theorem 4. *Suppose that assumptions $A1 - A4$ hold, and that bidders truthfully report their demand. Let $\boldsymbol{p}(T)$ be a price vector the EAPAP reaches at $T \in [0, \infty)$. Then, the EAPAP ends at $\boldsymbol{p}(T)$ in finite* steps *if and only if $\boldsymbol{p}(T)$ is a supporting price vector.*

Proof. Define a new economy by adding a fictitious bidder 0 with the consumption set $X_0 = \{\boldsymbol{x_0} \in \mathbb{Z}^K : \boldsymbol{0} \le \boldsymbol{x_0} \le \boldsymbol{S}\}$, and utility function $U_0(\boldsymbol{x_0}) = \boldsymbol{p}(T) \cdot \boldsymbol{x_0}$ to the economy. So, the indirect utility function of bidder 0 is $V_0(\boldsymbol{p}) = ((\boldsymbol{p}(T) - \boldsymbol{p}) \vee \boldsymbol{0}) \cdot \boldsymbol{S}$. Observe that $\boldsymbol{p}(T)$ is a Walrasian equilibrium price of the economy with bidder 0 if and only if $\boldsymbol{p}(T)$ is a supporting price vector of the economy without bidder 0. The Lyapunov function for the economy with bidder 0 is $\widetilde{L}(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{S} + \sum_{i=0}^{n} V_i(\boldsymbol{p})$, and

$$\begin{aligned} \widetilde{L}(\boldsymbol{p}(T)) &= \boldsymbol{p}(T) \cdot \boldsymbol{S} + \sum_{i=0}^{n} V_i(\boldsymbol{p}(T)) \\ &= \boldsymbol{p}(T) \cdot \boldsymbol{S} + \sum_{i=1}^{n} V_i(\boldsymbol{p}(T)). \end{aligned} \tag{69}$$

Since $V_i(\boldsymbol{p}) \ge V_i(\boldsymbol{p}(T) \vee \boldsymbol{p})$ for any $i \in N \cup \{0\}$, $\widetilde{L}(\boldsymbol{p}) \ge \widetilde{L}(\boldsymbol{p}(T) \vee \boldsymbol{p})$ for any $\boldsymbol{p} \in \mathbb{R}^K_{++}$. Now suppose that the price adjustment procedure in the economy with bidder 0 terminates at $\boldsymbol{p}(T)$. Then, $\boldsymbol{p}(T)$ minimizes $\widetilde{L}(\cdot)$. Suppose on the contrary that there exists $\boldsymbol{p} \in \mathbb{R}^K_{++}$, $\boldsymbol{p} \ne \boldsymbol{p}(T)$, such that $\widetilde{L}(\boldsymbol{p}) < \widetilde{L}(\boldsymbol{p}(T))$. Observe that $\widetilde{L}(\boldsymbol{p} \vee \boldsymbol{p}(T)) < \widetilde{L}(\boldsymbol{p}(T))$. So, for any $\delta(\boldsymbol{p}) \in \mathbb{R}, \delta(\boldsymbol{p}) > 0$, there exists $\boldsymbol{p'} \in \{\boldsymbol{p}(T) + \boldsymbol{\Delta} : \boldsymbol{0} \lneq \boldsymbol{\Delta} \le \boldsymbol{\delta}(\boldsymbol{p})^K\}$ such that $\boldsymbol{p'}$ is a strict convex combination of $\boldsymbol{p} \vee \boldsymbol{p}(T)$ and $\boldsymbol{p}(T)$. As, by Lemma 2, $\widetilde{L}(\cdot)$ is a convex function, $\widetilde{L}(\boldsymbol{p'}) < \widetilde{L}(\boldsymbol{p}(T))$. This contradicts to the assumption that the algorithm terminated at $\boldsymbol{p}(T)$.

Since $\boldsymbol{p}(T)$ is a minimizer of $\widetilde{L}(\cdot)$, applying Proposition 1 to $\widetilde{L}(\cdot)$, $\boldsymbol{p}(T)$ is a Walrasian equilibrium price of the economy with bidder 0. Therefore $\boldsymbol{p}(T)$ is a supporting price of the economy without bidder 0. By Theorem 1, the price adjustment procedure converges in finite *steps*. ☐

The extended global Walrasian tâtonnement consists of two main components, as explained in Section 3: The EAPAP and the EDPAP. The following Theorem shows the finite-*step* convergence of the extended global Walrasian tâtonnement to a Walrasian equilibrium price vector starting from an arbitrary initial price vector.

Theorem 5. *Suppose that assumptions $A1 - A4$ hold, and that bidders report truthfully. Then, starting from any initial price vector $\boldsymbol{p}(0) \in \mathbb{R}^K_+$, the extended global Walrasian tâtonnement reaches a Walrasian equilibrium price vector $\boldsymbol{p^*} \in \mathbb{R}^K_+$ in finite* steps.

Proof. The extended global Walrasian tâtonnement converges in finite *steps* terminates only when the EAPAP terminates, and by Proposition 4, $\boldsymbol{p}(T) \ge$

$\underline{\boldsymbol{p}}$. Suppose on the contrary that $\boldsymbol{p}(T)$ is not a Walrasian price vector. Then, by Proposition 1, $L(\underline{\boldsymbol{p}}) < L(\boldsymbol{p}(T))$. For any $\delta(\boldsymbol{p}) \in \mathbb{R}, \delta(\boldsymbol{p}) > 0$, there exists $\boldsymbol{p'}$, a strict convex combination of $\boldsymbol{p}(T)$ and $\underline{\boldsymbol{p}}$ such that $\boldsymbol{p'} \in \{\boldsymbol{p}(T) - \boldsymbol{\Delta} : \boldsymbol{0} \lneq \boldsymbol{\Delta} \leq \boldsymbol{\delta}(\boldsymbol{p})^K\}$. By Lemma 2 $L(\cdot)$ is a convex function. So $L(\boldsymbol{p'}) < L(\boldsymbol{p}(T))$. Choose $\delta(\boldsymbol{p})$ from the definition of $\boldsymbol{p_-}$ where $\boldsymbol{p_-} = \boldsymbol{p} - \boldsymbol{\delta}(\boldsymbol{p})^{E_-(\boldsymbol{p})}$ for some $E_-(\boldsymbol{p}) \subset K$. Since $\boldsymbol{p_-}$ is a maximal minimizer, $L(\boldsymbol{p_-}) \leq L(\boldsymbol{p'})$ implying $L(\boldsymbol{p_-}) < L(\boldsymbol{p'})$. Hence, the EAPAP cannot terminate at T, a contradiction.

As X_i is finite for each i, and since N is finite, there exists a K dimensional cube in which $\boldsymbol{p^*}$ lies such that starting from any price vector, $\boldsymbol{p}$, in this cube, for each $E \subset K$, there is no bidder who adds a bundle to his demand set at any price reached in the cube as prices of goods in E are all increased (or decreased). Therefore, when prices reach this cube in the price adjustment, they can change direction finitely many times. So, the extended global Walrasian tâtonnement reaches a Walrasian price vector in finite steps.

□

5 Results

Now the results of Ausubel [2] can be shown to hold in the extended environment. Note that the proofs of the following Theorems are the same as those given in Ausubel [2].

The *modified VCG mechanism with price of* $\boldsymbol{p}(0)$ is the following procedure: Each bidder i reports a function of valuation of objects, $U_i : X_i \to \mathbb{R}$, the nonlinear component of the utility function $u_i(\cdot)$, to the auctioneer, and the auctioneer assigns a consumption bundle, $\boldsymbol{x_i^*}$ to each bidder i and charges a payment of $\boldsymbol{y_i^*} = U_i(\boldsymbol{x_i^*}) - W^{**} + W^{**}_{-i}$, where

$$(\boldsymbol{x_i^*})_{i \in N} \in \arg\max \Big\{ \sum_{i \in N} (U_i(\boldsymbol{x_i}) - \boldsymbol{p}(0)) : \boldsymbol{x_i} \in X_i \text{ and } \sum_{i \in N} \boldsymbol{x_i} = \boldsymbol{S} \Big\}, \quad (70)$$

$$W^{**} = \max \Big\{ \sum_{i \in N} (U_i(\boldsymbol{x_i}) - \boldsymbol{p}(0)) : \boldsymbol{x_i} \in X_i \text{ and } \sum_{i \in N} \boldsymbol{x_i} = \boldsymbol{S} \Big\}, \quad (71)$$

and

$$W^{**}_{-i} = \max \Big\{ \sum_{\substack{j \in N \\ j \neq i}} (U_j(\boldsymbol{x_j}) - \boldsymbol{p}(0)) : \boldsymbol{x_j} \in X_j \Big\}. \quad (72)$$

The strategy profile $(\boldsymbol{\sigma_i})_{i \in N}$ constitutes an *ex post perfect* equilibrium if for every time t, following any history H_i^t, and for every realization

$(u_i(\cdot)_i)_{i\in N}$ of private information, the profile of continuation strategies $(\boldsymbol{\sigma_i}(\cdot,\cdot|t,H_i^t,u_i(\cdot)))_{i\in N}$ constitutes a *Nash* equilibrium of the game in which the realization of $(u_i(\cdot)_i)_{i\in N}$ is common knowledge.

The *parallel auction game* starts with any initial price $\boldsymbol{p}(0)\in\mathbb{R}^K$, and for each bidder $i\in N$ the following auction is run: Price $\boldsymbol{p}(t)$ is adjusted according to the extended global Walrasian tâtonnement for the subeconomy $N\setminus\{i\}$ until a price $\boldsymbol{p}_{-i}$ is reached at which the market clears without bidder i. After completing these N auctions, the auctioneer chooses an arbitrary bidder $j\in N$, and starting from $\boldsymbol{p}_{-j}$, runs the Ausubel auction. Suppose that it terminates at time $T\in[0,\infty)$. Each bidder $i\in N$ receives the market clearing objects in $Q_i(\boldsymbol{p}(T))$. The payment of bidder $j\in N$ is computed as the payment made along the curve from $\boldsymbol{p}_{-j}$ to $\boldsymbol{p}(T)$. The payment of bidder $i\in N, i\neq j$ is computed as the sum of payments along curves from $\boldsymbol{p}_{-i}$ to $\boldsymbol{p}(0)$, $\boldsymbol{p}(0)$ to $\boldsymbol{p}_{-j}$, and $\boldsymbol{p}_{-j}$ to $\boldsymbol{p}(T)$.

Theorem 6. *Suppose that the assumptions A.*1*, A.*2*, A.*3 *and A.*4 *hold. At any price vector* $\boldsymbol{p}$*, if every other bidder* $j\neq i$ *bids sincerely according to utility function* $u_j(\cdot)$*, then bidder* i *maximizes his payoff by bidding sincerely, and this maximizes social surplus* $U_i(\cdot)+\sum_{\substack{j\in N\\ j\neq i}}U_j(\cdot)$ *over all allocations.*

Theorem 7. *Suppose that the assumptions A.*1*, A.*2*, A.*3 *and A.*4 *hold. Assuming that the participation is mandatory:*

1. *Sincere bidding by every bidder is an ex post perfect equilibrium of the auction game;*

2. *Starting from an arbitrary initial price vector* $\boldsymbol{p}(0)\in\mathbb{R}^K_+$*, if bidders bid sincerely, then prices converge to a Walrasian equilibrium price vector in finite* steps*; and*

3. *Starting from an arbitrary initial price vector* $\boldsymbol{p}(0)\in\mathbb{R}^K_+$*, if bidders bid sincerely, then the outcome is that of the modified VCG mechanism with price of* $\boldsymbol{p}(0)$*.*

Theorem 8. *Suppose that the assumptions A.*1*, A.*2*, A.*3 *and A.*4 *hold. Let the initial price* $\boldsymbol{p}(0)\in\mathbb{R}^K_+$ *be a Walrasian equilibrium price vector of the market without bidder* i*. If each bidder* $j\neq i$ *bids sincerely, then sincere bidding is bidder* i*'s best response and bidder* i *receives his VCG payoff.*

Theorem 9. *Suppose that the assumptions A.*1*, A.*2*, A.*3 *and A.*4 *hold. Starting from any arbitrary initial price vector* $\boldsymbol{p}(0)\in\mathbb{R}^K_+$*, sincere bidding by every bidder constitute an* ex post perfect *equilibrium of the parallel auction* *game. Prices converge to a Walrasian equilibrium price vector in finite* steps, *and each bidder receives his VCG payoff.*

6 Conclusion

In this paper, I extended the ascending and the descending price adjustment procedures proposed in Ausubel [2] to real-valued quasilinear utility functions. I show that these extended procedures converge to a Walrasian equilibrium price vector in finite *steps*. With these extensions, I extend the global Walrasian tâtonnement, Ausubel [2], to real-valued quasilinear utility functions and show its convergence to a Walrasian equilibrium price vector in finite *steps*. Unlike the global Walrasian tâtonnement in Ausubel [2], the extended global Walrasian tâtonnement does not require any information on bidders' when bidders' values for bundles real. I also show that Ausubel's auction for discrete heterogeneous goods has sincere bidding as an efficient equilibrium when the extended global Walrasian tâtonnement is used to adjust prices. Furthermore, if n price trajectories are run, then the VCG mechanism is implemented.

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