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EERI Research Paper Series No 4/2008

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An Efficient Auction for Heterogeneous Discrete Goods When Preferences are Separable

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March 16, 2008

Draft

Abstract

I design a dynamic auction mechanism for heterogeneous discrete goods when bidders' for goods are interdependent. Each good is allocated separately by Perry and Reny's mechanism for homogeneous goods, Perry and Reny [3]. I show that if agents' utility functions are additively separable in goods, in addition to the assumptions in Perry and Reny [3], then "equilibrium strategies" constitute an efficient equilibrium outcome.

1 Introduction

Auctioning of multiple goods gained attention recently, especially after FCC started auctioning communication bands to wireless companies in 1994. Designing an efficient dynamic auction mechanism for multiple goods when values are interdependent is one of the challenging questions of auction theory. Perry and Reny [3] proposed an efficient dynamic auction for discrete homogeneous goods in interdependent-values environment, but the case of heterogeneous discrete goods is still an open question. Maskin [2] highlights the importance of such mechanism.

*I would like to thank my advisors Ket Richter and Jan Werner for their continuous support, patience, and effort for my education. Their guidance has been enlightening. I also would like to thank Pat Bajari and Rajdeep Singh for their useful comments. All errors are mine.

In this paper, I design an auction mechanism for heterogeneous discrete goods in interdependent-values environment using Perry and Reny's homogeneous goods auction mechanism, Perry and Reny [3], for each good separately. The mechanism designed has an *ex post* efficient equilibrium outcome. This paper is a step closer towards a mechanism for a more general environment.

Perry and Reny's auction mechanism, Perry and Reny [3], is a generalization of Ausubel's auction for homogeneous goods, Ausubel [1], to interdependent values. In Ausubel's auction for homogeneous goods, the price starts at 0 and increased by 1 at each period. Bidders submit their demand at the current price in the auction. As the price rises, a bidder "clinches" a unit if the rest of the bidders stop demanding that unit. He pays the current price for that unit he clinches. If there is still excess demand in the market then the price continues to rise. The auction stops whenever the market clears. In Perry and Renny [3], each bidder privately receives a one dimensional signal. A bidder's utility from consumption depends on the signals of all bidders. Since bidders do not know each other's private signal, they do not know marginal value of a unit of the good. Hence, they do not know, as the price rises, when to decrease the number of units they demand. Perry and Reny [3] define "equilibrium strategies" using marginal value curves of bidders. Bidders use these strategies to determine prices at which to drop units from their demand. Perry and Reny [3] show that these strategies constitute an *ex post* efficient equilibrium. Bidders are assumed to have utility functions quasilinear in money, and wealths large enough so that their budget constraints do not bind. A bidder's marginal value from a unit is assumed to be increasing in the profile of signals of all bidders, and strictly increasing in bidder's own signal. Bidders are also assumed to have marginal values decreasing in number of units consumed. Marginal values of bidders are also assumed to satisfy a single-crossing property. A continuity assumption on the prior distribution of the joint signal space rules out the possibility of two distinct bidders' marginal values of units to be equal.

An efficient dynamic auction mechanism for heterogeneous goods when values are interdependent is still a missing piece in the literature. This paper tries to close this gap in a special environment. Perry and Reny [3] auction is used to allocate each good type separately.

I show that since preferences are additively separable in goods, efficient allocation of a good does not depend on the allocation of any other good. Moreover, I also show that because budget constraints do not bind, preferences are quasilinear in payments, and preferences are additively separable in goods, at any price vector, the amount of any good in an agent's utility maximizing consumption bundle does not depend on prices of other goods. Hence, if agents' utility functions are assumed to be additively separable in goods in addition to the assumptions in Perry and Reny [3], then strategies constructed from "equilibrium strategies" of Perry and Reny [3] for each good constitute an *ex post* efficient equilibrium.

In Section 2, the assumptions of the model is given. In Section 3, the main results are presented. Finally, in Section 4, Perry and Reny auction is given and implications of their results and the results of the previous section are discussed.

2 The Model

There are $K = \{1, \ldots, K\}$ goods and $N = \{1, \ldots, N\}$ bidders. Total supply of each good $k \in K$ is $L_k \in \mathbb{Z}_{++}$ units. Each bidder $i \in N$ receives a one-dimensional private signal $s_i \in S_i \subseteq \mathbb{R}$. His value for bundle $\boldsymbol{x^i} = (x^i_k)_{k \in K} \in X_i \subset \mathbb{Z}^K$ depends on the profile of signals $\boldsymbol{s} = (s_i)_{i \in N} \in S = \prod_{i \in N} S_i$ and is denoted by $u^i(\boldsymbol{x^i}; \boldsymbol{s})$. Agent $i \in N$'s utility function is quasilinear in money, t^i

$$U^i(\boldsymbol{x^i}, t^i; \boldsymbol{s}) = u^i(\boldsymbol{x^i}; \boldsymbol{s}) + t^i, \tag{1}$$

So

$$U^i(\boldsymbol{x^i}, \overline{m}^i - y^i; \boldsymbol{s}) = u^i(\boldsymbol{x^i}; \boldsymbol{s}) + \overline{m}^i - y^i \tag{2}$$

where y^i is the amount of expense i makes and $\overline{m}^i$ is the initial wealth of bidder i. Initial wealths are so large that budget constraints do not bind. Therefore, $\overline{m}^i$ will be dropped without loss of generality. Utility that bidder i receives from consuming bundle $\boldsymbol{x^i}$ when prices of goods are given by the price vector $\boldsymbol{p} = (p_k)_{k \in K} \in \mathbb{R}^K_+$ is $u^i(\boldsymbol{x^i}; \boldsymbol{s}) - \boldsymbol{p} \cdot \boldsymbol{x^i}$.

The following assumptions are made:

A.0 Separable Utilities For each $i \in N$

$$U^i(\boldsymbol{x^i}, t^i; \boldsymbol{s}) = u^i(\boldsymbol{x^i}; \boldsymbol{s}) + t^i = \sum_{k=1}^{K} u^i_k(x^i_k; \boldsymbol{s}) + t^i \tag{3}$$

where x^i_k is the amount of good k agent i consumes.

Assumption **A.0** implies that for each agent $i \in N$, the marginal value of l^{th} unit of good k for agent i is

$$v^i_{kl}(\boldsymbol{s}) = u^i_k(l; \boldsymbol{s}) - u^i_k(l-1; \boldsymbol{s}). \tag{4}$$

Each bidder i is assumed to know only his own signal s_i.

A.1 Diminishing marginal utility For each $i \in N$, each $\boldsymbol{s} \in S$, each $k \in K$, and each $l \in \{1, 2, \ldots, L_k - 1\}$,

$$v^i_{kl}(\boldsymbol{s}) \geq v^i_{kl+1}(\boldsymbol{s}). \tag{5}$$

A.2 Monotonicity in signals For each $i \in N$, each $\boldsymbol{s} \in S$, each $k \in K$, and each $l \in \{1, 2, \ldots, L_k\}$, $v^i_{kl}(\boldsymbol{s})$ is weakly increasing in $\boldsymbol{s}$ and strictly increasing in s_i.

A.3 Single-crossing property For each $i, j \in N, i \neq j$, each $\boldsymbol{s} \in S$, each $k \in K$, and each $l, m \in \{1, 2, \ldots, L_k\}$, if $v^i_{kl}(\boldsymbol{s}) \geq v^j_{km}(\boldsymbol{s})$, then the inequality is strict when s_i rises or s_j falls, and all other components of $\boldsymbol{s}$ remain unchanged.

An additional assumption,**A.4**, in Perry and Reny [3] on the distribution of signal spaces, and assumption **A.3** above imply that for any bidder, his marginal value for a unit is different from any other bidder's marginal value for any unit.

In an auction game, a profile of strategies is an *ex post* equilibrium if for any profile of private signals of bidders, these strategies constitute a *Nash* equilibrium of the game in which this profile of private signals is commonly known.

3 Designing An Efficient Auction Mechanism for Heterogeneous Goods

In this section, there are two results. Lemma 1 shows that under separability of preferences, if an efficient homogeneous goods auction mechanism is used to allocate each good separately, then the resulting allocation is efficient. This is true whether these auctions are run simultaneously or sequentially. Lemma 2, on the other hand, shows that if preferences are separable, quasi-linear in payments, and the budget constraints do not bind, then the strategy a bidder plays in homogeneous good auction mechanism of each good does not depend on the outcome of the homogeneous good mechanism of the other goods.

Let $\boldsymbol{x_k} = (x^i_k)_{i \in N}$ stand for the vector consisting of amounts x^i_k of good $k \in K$ each agent $i \in N$ consumes.

Lemma 1. *[The Separability of Efficient Allocation] Assume that textbfA.0 holds. Then, The distribution of each good in an efficient allocation is independent of distributions of the rest of the goods in that efficient allocation, i.e.,*

$$\boldsymbol{x} = (\boldsymbol{x_k})_{k \in K} \in \underset{\boldsymbol{x}=(\boldsymbol{x_k})_{k\in K} \in \mathbb{Z}_+^{NK}}{\arg\max} \sum_{i \in N} \sum_{k \in K} u_k^i(x_k^i; \boldsymbol{s}) \tag{6}$$

$$\textit{such that for all } k \in K, \sum_{i \in N} x_k^i = L_k \tag{7}$$

if and only if

$$\textit{for all } k \in K, \boldsymbol{x_k} \in \underset{\boldsymbol{x_k}=(x_k^i)_{i\in N} \in \mathbb{Z}_+^{N}}{\arg\max} \sum_{i \in N} u_k^i(x_k^i; \boldsymbol{s}) \tag{8}$$

$$\textit{such that } \sum_{i \in N} x_k^i = L_k. \tag{9}$$

Proof. Suppose on the contrary that there exists

$$\boldsymbol{x} = (\boldsymbol{x_k})_{k \in K} \in \underset{\boldsymbol{x}=(\boldsymbol{x_k})_{k\in K} \in \mathbb{Z}_+^{NK}}{\arg\max} \sum_{i \in N} \sum_{k \in K} u_k^i(x_k^i; \boldsymbol{s}) \tag{10}$$

$$\text{such that for all } k \in K, \sum_{i \in N} x_k^i = L_k \tag{11}$$

such that for some $k \in K$

$$\boldsymbol{x_k} \notin \underset{\boldsymbol{x_k}=(x_k^i)_{i\in N} \in \mathbb{Z}_+^{N}}{\arg\max} \sum_{i \in N} u_k^i(x_k^i; \boldsymbol{s}) \tag{12}$$

$$\text{such that } \sum_{i \in N} x_k^i = L_k. \tag{13}$$

Then there exists $\tilde{\boldsymbol{x}}_{\boldsymbol{k}} = (\tilde{x}_k^i)_{i \in N}$, $\sum_{i \in N} \tilde{x}_k^i = L_k$ such that

$$\sum_{i \in N} u_k^i(\tilde{x}_k^i; \boldsymbol{s}) > \sum_{i \in N} u_k^i(x_k^i; \boldsymbol{s}). \tag{14}$$

For all $k' \in K, k' \neq k$, construct $\tilde{\boldsymbol{x}}_{\boldsymbol{k'}} = \boldsymbol{x_{k'}}$. Then for all $k' \in K, k' \neq k$,

$$\sum_{i \in N} u_{k'}^i(\tilde{x}_{k'}^i; \boldsymbol{s}) = \sum_{i \in N} u_{k'}^i(x_{k'}^i; \boldsymbol{s}). \tag{15}$$

These imply that

$$\sum_{i \in N} \sum_{k \in K} u_k^i(\tilde{x}_k^i; \boldsymbol{s}) > \sum_{i \in N} \sum_{k \in K} u_k^i(x_k^i; \boldsymbol{s}), \tag{16}$$

a contradiction to Expression 10.

Now suppose on the contrary that there exists $\boldsymbol{x} = (\boldsymbol{x_k})_{k \in K}$ such that

$$\text{for all } k \in K, \boldsymbol{x_k} \in \underset{\boldsymbol{x_k}=(x_k^i)_{i\in N} \in \mathbb{Z}_+^{N}}{\arg\max} \sum_{i \in N} u_k^i(x_k^i; \boldsymbol{s}) \tag{17}$$

$$\text{such that } \sum_{i \in N} x_k^i = L_k \tag{18}$$

and there exists $\tilde{\boldsymbol{x}}$, for all $k \in K, \sum_{i\in N} \tilde{x}^i_k = L_k$ such that

$$\sum_{i\in N}\sum_{k\in K} u^i_k(\tilde{x}^i_k;\boldsymbol{s}) > \sum_{i\in N}\sum_{k\in K} u^i_k(x^i_k;\boldsymbol{s}). \tag{19}$$

This implies that there exists $k \in K$ such that

$$\sum_{i\in N} u^i_k(\tilde{x}^i_k;\boldsymbol{s}) > \sum_{i\in N} u^i_k(x^i_k;\boldsymbol{s}), \tag{20}$$

a contradiction to Expression 17. □

Lemma 2. *[The Separability of Agent's Problem] Assume that **A.0** holds, and that preferences are quasilinear in payments and the budget constraints do not bind. Then, at each price vector $\boldsymbol{p} = (p_k)_{k\in K}$ the utility maximizing amount of each good is independent of the prices of the other goods, i.e.,*

$$x^i_k \in \arg\max u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k \text{ for each } k \in K \tag{21}$$

if and only if

$$\boldsymbol{x^i} = (x^i_k)_{k\in K} \in \arg\max \sum_{k\in K}\left[u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k\right]. \tag{22}$$

Proof. Suppose on the contrary that there exists $\boldsymbol{x^i} = (x^i_k)_{k\in K}$

$$\text{for all } k \in K, x^i_k \in \arg\max u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k \tag{23}$$

and

$$\tilde{\boldsymbol{x}}^{\boldsymbol{i}} = (\tilde{x}^i_k)_{k\in K} \tag{24}$$

such that

$$\sum_{k\in K}\left[u^i_k(\tilde{x}^i_k;\boldsymbol{s}) - p_k\tilde{x}^i_k\right] > \sum_{k\in K}\left[u^i_k(\tilde{x}^i_k;\boldsymbol{s}) - p_k\tilde{x}^i_k\right]. \tag{25}$$

This implies that there exists $k \in K$ such that

$$u^i_k(\tilde{x}^i_k;\boldsymbol{s}) - p_k\tilde{x}^i_k > u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k, \tag{26}$$

a contradiction to Expression 23.

Now suppose on the contrary that there exists

$$\boldsymbol{x^i} = (x^i_k)_{k\in K} \in \arg\max \sum_{k\in K}\left[u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k\right] \tag{27}$$

such that for some $k \in K$

$$x^i_k \notin \arg\max u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k. \tag{28}$$

This implies that there exists $\tilde{x}^i_k$ such that

$$u^i_k(\tilde{x}^i_k;\boldsymbol{s}) - p_k\tilde{x}^i_k > u^i_k(x^i_k;\boldsymbol{s}) - p_k x^i_k. \tag{29}$$

Construct $\tilde{x}^i_{k'}$ for all $k' \in K, k' \neq k$ such that

$$\tilde{x}^i_{k'} = x^i_{k'}. \tag{30}$$

Then

$$\sum_{k \in K} \left[u^i_k(\tilde{x}^i_k; \boldsymbol{s}) - p_k \tilde{x}^i_k \right] > \sum_{k \in K} \left[u^i_k(x^i_k; \boldsymbol{s}) - p_k x^i_k \right], \tag{31}$$

a contradiction to Expression 27.

□

4 The Perry-Reny Mechanism and Ausubel's Auction

In the homogeneous good Ausubel auction [1], the auctioneer raises the price of the good by 1 at each period, and bidders report their demand sets at the current price. At any price, a bidder "clinches" a unit if the rest of the bidders drop that unit from their demand at the current price. He then pays the price at which he clinches this unit. The auction stops if there is no excess demand anymore at the current price. Perry and Reny [3] introduce "equilibrium strategies" to be played by bidders. As bidders' values for units depend on the private signals of all bidders, no bidder knows his value for these units. Therefore, they do not know at what price to lower the quantity they demand. "Equilibrium strategies" guide bidders throughout the auction for prices at which they should lower their demands for goods as the price of the good increases. These strategies use demand-reduction prices, which are constructed using bidders' marginal value functions. Note that since each bidder receives a private signal, he knows his own marginal value for a unit as a function of the signals of the other bidders. Whenever the price reaches a bidder's demand-reduction price for a unit, the bidder drops demanding this unit and lowers the quantity he demands. Perry and Reny [3] show that when there are two bidders, these "equilibrium strategies" constitute an *ex post* efficient equilibrium in Ausubel's auction [1].

4.1 Two Bidders

Suppose that $N = 2$ and instead of the assumption **A.3**, the following holds:

A.3′ Assume that $A.3$ holds and for any $k \in K$, $l, 1 \leq l \leq L_k$, $\boldsymbol{s} = (s_1, s_2) \in S$ there exists a pair of signals α, β such that

$$\begin{aligned} v^1_{kl}(s_1, \alpha) &= v^2_{kL_k - l}(s_1, \alpha) \\ v^2_{kl}(\beta, s_2) &= v^1_{kL_k - l}(\beta, s_2). \end{aligned} \tag{32}$$

Bidder 1's demand-reduction price for l^{th} unit of good k is defined as

$$p_{kl}^1(s_1) = v_{kl}^1(s_1, \alpha). \tag{33}$$

Similarly, bidder 2's demand-reduction price for l^{th} unit of good k is defined as

$$p_{kl}^2(s_2) = v_{kl}^2(\beta, s_2). \tag{34}$$

So, if the price of good k, p_k in the auction reaches $p_{kl}^1(s_1)$, then bidder 1 lowers his demand for good k from l to $l-1$. Similarly, if the price of good k, p_k in the auction reaches $p_{kl}^2(s_2)$, then bidder 2 lowers his demand for good k from l to $l-1$. Demand-reduction prices have two important properties, as shown in Perry and Reny [3]:

Monotonicity of demand-reduction prices The demand-reduction prices are monotone decreasing in units of goods: For all $i \in N$, all $s_i \in S_i$, all $k \in K$, and all $l \in \{1, 2, \dots, L_k - 1\}$,

$$p_{kl}^i(s_i) \geq p_{kl+1}^i(s_i). \tag{35}$$

Separation of values The demand-reduction prices separate agents according to their values, i.e., for all $i, j \in N$, all $\boldsymbol{s} \in S$, all $k \in K$, and all $l \in \{1, 2, \dots, L_k\}$,

$$p_{kl}^i(s_i) \geq p_{kL_k-l+1}^j(s_j)$$

if and only if

$$p_{kl}^i(s_i) \geq v_{kl}^i(s_i) \geq v_{kL_k-l+1}^j(s_j) \geq p_{kL_k-l+1}^j(s_j).$$

Proposition 1. *Suppose that* $\mathbf{A.1}, \mathbf{A.2}$, *and* $\mathbf{A.3'}$ *hold, and that* $N = 2$. *Use of demand-reduction prices by bidders constitute an ex post efficient equilibrium of Ausubel's auction.*

Proof. The proof follows from Lemmas 1 and 2 above and Proposition 4.1 in Perry and Reny [3]. □

4.2 General Case

In the general Perry and Reny [3] auction mechanism, agents use demand-reduction prices to change their demand but they reveal their demands against every other bidder in the auction. So if there are N bidders in the auction, every agent will reveal a profile of $N-1$ demands, one for every other bidder. It is said that bidders place *directed demands*. Moreover, unlike the Ausubel auction, bidders can increase their demands at various stages of the auction, and the price is not allowed to decrease.

The main theorem of Perry and Reny [3] holds for each type of good. So, under assumptions **A.0-A.3**, by Lemmas 1 and 2, the mechanism consisting of Perry-Reny mechanisms for each type of good has an *ex post* efficient equilibrium.

5 Conclusion

This paper presented an auction design for heterogeneous goods when bidders' values for goods are interdependent on each other's private signals. This is achieved by using Perry and Reny's auction mechanism for homogeneous goods, Perry and Reny [3], for each good separately. The auction mechanism introduced here consists of Perry-Reny mechanisms for each good, and it is shown to have an ex post efficient equilibrium when there are two bidders. An analogous result holds for more bidders when Perry-Reny homogeneous good mechanism for 3 or more bidders is used for each good separately.

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