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The principle has useful production theory applications: in supply chain modelling, in the determination of the optimal depth of production processes. Growth models are extended to allow for the hypothesis.

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ABSTRACT

On Depth and Retrospect: "I Forget, and Forgive – but I Discount."

The discounting of future felicity flows transposes to the intertemporal optimization context the assumption of interest-bearing wealth or savings. The validity of the hypothesis has been challenged by several empirical (ir)regularities and by the theoretical implications for human decision processing. In particular, it implies a very special weight of past decisions on current welfare prospects, which appears largely inconsistent with forgetfulness – even if not with learning – and memory effects, often stressed or embedded in behavioral science studies. In this article, we explore the modifications induced by generalizing the typical welfare function in order to accommodate such retrospective influences. The idea is simple – and can be thought inspired in felicity functions encompassing habit formation: to allow for accumulated welfare – of hypothetically "compounded" but also depreciating past-to-current felicity streams – to affect the periodic utility function – which therefore enjoy some durable good properties. Sensitivity of the Ramsey optimal path to the new formulation is also inspected.

The mathematical principle has useful production theory applications: in supply chain modelling. Then the optimal depth of a production process stems from a standard problem that now also embeds delay evaluation – discounting; a rationale for a particular pattern of the term structure of interest rates was also forwarded. Growth – general equilibrium - models are extended to allow for the hypothesis.

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On Depth and Retrospect: “I Forget, and Forgive – but I Discount”

“I make known the end from the beginning, from ancient times, what is still to come.” In *Isaiah*, 46: 10.

“If only you had paid attention to my commands, your peace would have been like a river, your righteousness like the waves of the sea.” In *Isaiah*, 48: 18.

“The memory of the [uncompromisingly] righteous is a blessing, but the name of the wicked shall rot.” In *Proverbs*, 10: 7.

“The foundation of the temple of the LORD was laid in the fourth year, in the month of Ziv. In the eleventh year in the month of Bul, the eighth month, the temple was finished in all its details according to its specifications. He had spent seven years building it. It took Solomon thirteen years, however, to complete the construction of his palace. He built the Palace of the Forest of Lebanon a hundred cubits long, (…)” In *I Kings*, 6: 37-38 and 7: 1-2.

Introduction

The cumulative discounting of future flows of periodic utility – felicity functions - has become a generally accepted hypothesis in intertemporal choice modelling. Its justification [1] is usually prospective: individuals praise the present more than the future, and the close future more than the distant one. It is the purpose of these lines to state, on the one hand, a backward-looking argument for its convenience in theoretical representations of multi-period lived agents, and on the other, to generalize the welfare maximand to incorporate retrospect influences echoing on current felicity. Some applications of the same mechanics to product-chain systems modelling are also illustrated.

The standard perspective stems from the arithmetic of a conventional intertemporal wealth constraint: if the future is discounted, the past is implicitly compounded and at the same rate. The hypothesis has been abandoned in several studies that tried to explain empirical anomalies, observed in asset markets or in behavioral experiments, incompatible with the principle. Those extensions took two different directions: incorporating re-enforcing effects of

[1] See Frederick, Loewenstein and O’Donoghue (2002) for a recent survey.

future decisions – that is the case of prospect theory; and of past practices – in which analysis of habit formation is an example. We follow the second branch of the literature [2].

The reason is simple: we want a filter of previous recollections to dim the marginal rate of substitution between past and current prospects. Therefore, we introduce cumulative past-to-current welfare as an argument of the current felicity (function) itself. Then, utility has a potential akin to that of a durable good – once materialized, it accrues to all future wealth. If allowed to depreciate, the weight of past well-being can reproduce forgetfulness. In growth terms, it is as if the passage of time allowed for a stock [3] (of welfare) to accumulate, accruing as an utility flow enhancing factor. In any case, the hypothesis would suggest felicity smoothing proneness and a complexification of inter-period links embedded in efficiency constraints

The principle has an obvious production application: in supply chain modelling: we can admit that the periodic production depends on the use of other inputs and on the product. Moreover, if such a product chain is attached to an intertemporal – firm or representative agent's - objective function, an interpretation of decision-making over a dynamically spanned production system is immediate. Optimal depth can then become endogenous. The issue has gained previous attention: Keren and Levhari (1983) model the organization of subsequent tasks that concur to a general input to a final production function; our approach generalizes and extends the argument, taking into account eventual discounting as depreciation along the dynamic process – and allow for other than intertemporally additive CRS of the periodic flow generation technology.

The juxtaposition of the supply chain to the Ramsey's structure and allowing for direct consumption of intermediate products was also inspected.

The exposition proceeds as follows: section 1 manipulates a standard intertemporal wealth constraint. The analysis is replicated for the case where retrospective traits are recognized in felicity functions in section 2. First-order conditions of the infinite-horizon consumption allocation problem are inspected in section 3. Trajectories of a standard Ramsey economy under the new preferences are studied in section 4. Section 5 explores potential applications to the modelling of vertical production arrangements. A final appraisal produces a concluding section.

[2] See section 5.2, specially 5.2.1, of Frederick, Loewenstein and O'Donoghue (2002).

[3] In habit formation literature, this stock is usually made of cumulative past consumption – see Becker and Murphy (1988), for example - of the addictive good.

1. Notation: The Effect of Standard Discounting [4]

Let an individual be born at time 0; U_t denotes his felicity in period t, and he is endowed with a stream U_t, t = 1,2,…,n. If he discounts the future at a periodic rate r – for simplicity, assume it constant -, at time 0, his prospective welfare is:

$$(1.1) \qquad W_0 = \sum_{t=1}^{\infty} \frac{U_t}{(1+r)^t} = \sum_{t=1}^{T} \frac{U_t}{(1+r)^t} + \frac{1}{(1+r)^T} \sum_{t=1}^{\infty} \frac{U_{T+t}}{(1+r)^t}$$

Standing at time T > 0, the individual's welfare prospects are $W_T = \sum_{t=1}^{\infty} \frac{U_{T+t}}{(1+r)^t}$; his evaluation, at that moment T, of all his life's initial potential is:

$$(1.2) \; W_0^T = W_0 (1+r)^T = \sum_{t=1}^{T} U_t (1+r)^{T-t} + \sum_{t=1}^{\infty} \frac{U_{T+t}}{(1+r)^t} = \sum_{t=1}^{T} U_t (1+r)^{T-t} + W_T =$$

$$\text{(letting } j = T - t\text{:)} \qquad = \sum_{j=0}^{T-1} U_{T-j} (1+r)^j + W_T$$

Then, $W_T = W_0^T - \sum_{j=0}^{T-1} U_{T-j} (1+r)^j$. All else (including total life's prospects) fixed, $-\frac{dW_T}{dU_{T-j}}$ is how much of an increase of current – time T's - prospects he would have to be (have been…) given to let go of one unit of a good memory of time (T – j). Using the last expression,

$$(1.3) \qquad -\frac{dW_T}{dU_{T-j}} = (1+r)^j$$

Such "price" of one unit of U_{T-j} for an individual located at T – the value at T of U_{T-j} – in terms of units of prospects at time T is larger than 1 and increases with j – with how distant in the past (T – j) is of T… So discounting mimics an increasing value and, therefore, effect (positive or negative: the argument would also apply to losses…) on current decisions, of previous remembrances as these become more distant in the past.

Also, as

[4] In appendix A, we derived the first results for variable periodic discount rates. The analysis becomes more complex but we derive similar views of the subject.

$$(1.4) \qquad W_0 = (1+r)^{-T} \sum_{j=0}^{T-1} U_{T-j}(1+r)^j + (1+r)^{-T} W_T = (1+r)^{-T'} \sum_{j=0}^{T'-1} U_{T'-j}(1+r)^j + (1+r)^{-T'} W_{T'}$$

the last equality suggests how accumulated prospects are relatively valued along lifetime indifference curves; if total wealth changes and that is not to affect U_t, $t \leq Max(T, T')$, the relative impact on accumulated prospects at the two points in time is

$$(1.5) \qquad \frac{\dfrac{dW_0}{dW_{T'}}}{\dfrac{dW_0}{dW_T}} = \frac{dW_T}{dW_{T'}} = (1+r)^{T-T'}$$

and we can write

$$(1.6) \qquad \frac{dW_T}{dW_{T-j}} = (1+r)^j = -\frac{dW_T}{dU_{T-j}}$$

In another angle, we note that:

$$(1.7) \quad (W_0^{\ T} - W_T) = \sum_{i=1}^{T} U_i (1+r)^{T-i} = (W_0^{\ T-1} - W_{T-1})(1+r) + U_T > (W_0^{\ T-1} - W_{T-1})$$

The aggregate representing recollections of past-to-present welfare at time T always increases with time. And:

$$(1.8) \quad \sum_{j=0}^{T-1} U_{T-j}(1+r)^j / \sum_{j=0}^{T-1-1} U_{T-1-j}(1+r)^j = (W_0^{\ T} - W_T) / (W_0^{\ T-1} - W_{T-1}) =$$
$$= [(W_0^{\ T-1} - W_{T-1})(1+r) + U_T] / (W_0^{\ T-1} - W_{T-1}) = (1+r) + U_T / (W_0^{\ T-1} - W_{T-1}) =$$
$$= (1+r) + U_T / [\sum_{j=0}^{T-1-1} U_{T-1-j}(1+r)^j]$$

If U_i grows at a constant rate g, so that $U_i = U_0 (1+g)^i$,

$$(1.9) \quad \sum_{j=0}^{T-1-1} U_{T-1-j}(1+r)^j = \sum_{j=0}^{T-2} U_0 (1+r)^j (1+g)^{T-1-j} = U_0 (1+g)^{T-1} \sum_{j=0}^{T-2} [(1+r)/(1+g)]^j = U_0 (1+g)^{T-1} \{1 - [(1+r)/(1+g)]^{T-1}\} / [1 - (1+r)/(1+g)] = U_T \{1 - [(1+r)/(1+g)]^{T-1}\} / (g - r)$$

Then (1.9) is equal to $(1+r) + (g-r) / \{1 - [(1+r)/(1+g)]^{T-1}\}$. As T goes to ∞, provided $r < g$, the rate of growth of past accumulated welfare, $(W_0^{\ T-1} - W_{T-1})$, also goes to g:

$$(1.10) \qquad \lim_{T\to\infty} (W_0{}^T - W_T) / (W_0{}^{T-1} - W_{T-1}) = 1 + g$$

A special case of constant U_t (g = 0) implies, after (1.1), that $U_t = r\,W_0$, for all t = 1,2,... Then, $\lim_{T\to\infty} (W_0{}^T - W_T) / (W_0{}^{T-1} - W_{T-1}) = 1$.

We conclude that utility discounting – as emergence of real interest-earning credits or loans - is compatible with (more or less exponentially…) increasing importance of past (first…) impressions with the time elapse, characterizing human behavior. We would be willing to pay a lot to change the past…

Proposition 1: Under conventional time discounting, the individual accepts a trade-off

1.1. of cumulative future prospects at a (future) moment in time with past felicity at a past moment in time, at a rate larger than 1 and increasing with the time distance as in (1.3) – the price at T of felicity at T - j.

1.2. of accumulated future prospects at a (future) moment in time with past ones at a rate larger than 1 and increasing with the time distance between them as in (1.5).

2. Retrospective Felicity Functions

. Suppose now that periodic felicity is a function of past experiences:

$$(2.1) \qquad U_t = U[\sum_{i=1}^{t} U_i\,(1 + r)^{t-i}] = U(W_0{}^t - W_t)$$

We have felicity functions that depend on accumulated past (and current) well-being, or of its memories. At each point in time, (1.1) and (1.2) – and (1.7), implying an increasing argument of U(.) over time - still apply, but now welfare is an implicit function: the current flow is itself a function of a stock backlog to which it contributes.

As we can write $U_t = U[\sum_{i=1}^{t-1} U_i\,(1 + r)^{t-i} + U_t] = U[(W_0{}^{t-1} - W_{t-1})\,(1 + r) + U_t]$, it is always possible to re-arrange and solve for $U_t = V[\sum_{i=1}^{t-1} U_i\,(1 + r)^{t-i}] = V[(1 + r) \sum_{i=1}^{t-1} U_i\,(1 + r)^{t-1-i}] = V[(W_0{}^{t-1} - W_{t-1})\,(1 + r)]$. Yet, if r goes to -1, (2.1) tends to a function of the standard form, of itself, U_t; in that case, a linear U(.) reverts welfare to the conventional discount aggregate of section 1.

(2.2) $W_T = W_0^T - \sum_{j=0}^{T-1} U[\sum_{i=1}^{T-j} U_i (1 + r)^{T-j-i}] (1 + r)^j = W_0^T - \sum_{j=0}^{T-1} U(W_0^{T-j} - W_{T-j}) (1 + r)^j$

Then, $-\frac{dW_T}{dU_{T-j}}$ is still $(1 + r)^j$. One can infer that

(2.3) $$\frac{dW_t}{dW_{t-j}}_{|W_0} = U'(W_0^{t-j} - W_{t-j}) (1 + r)^j$$

that measures how much of current prospects would one be willing to trade for one unit of prospects j periods ago.

Now

(2.4) $$W_0 = (1 + r)^{-T} \sum_{j=0}^{T-1} U(W_0^{T-j} - W_{T-j}) (1 + r)^j + (1 + r)^{-T} W_T = (1 + r)^{-T'} \sum_{j=0}^{T'-1} U(W_0^{T'-j} - W_{T'-j}) (1 + r)^j + (1 + r)^{-T'} W_{T'}$$

Then, for T > T',

(2.5) $$\frac{\frac{dW_0}{dW_{T'}}}{\frac{dW_0}{dW_T}} = \frac{dW_T}{dW_{T'}} = (1 + r)^{T-T'} [1 - U'(W_0^{T'} - W_{T'})]$$

Hence,

(2.6) $$\frac{dW_t}{dW_{t-j}} = [1 - U'(W_0^{t-j} - W_{t-j})] (1 + r)^j$$

If U'(.) = 0, we return to the standard case; but not otherwise. And as U'(.) > 0, the trade-off is smaller than in the standard case. Still, provided U'(.) < 1 – a reasonable assumption, implying one unit of "memories" – "echoes" of past to present welfare - are worth less than one unit of current felicity – the trade-off is still positive.

(2.7) $$\frac{d\left(\frac{dW_t}{dW_{t-j}}\right)}{dj} = \ln(1 + r) [1 - U'(W_0^{t-j} - W_{t-j})] (1 + r)^j - U''(W_0^{t-j} - W_{t-j}) (1 + r)^j \frac{d\left(W_0^{t-j} - W_{t-j}\right)}{dj}$$

$\frac{d\left(W_0^{t-j} - W_{t-j}\right)}{dj} \approx \sum_{i=1}^{t-j-1} U_i (1 + r)^{t-j-1-i} - \sum_{i=1}^{t-j} U_i (1 + r)^{t-j-i} = - U_{t-j} + [\sum_{i=1}^{t-j-1} U_i (1 + r)^{t-j-1-i}] (1 - 1 - r) = - U_{t-j} - r [W_0^{t-j-1} - W_{t-j-1}]$. Then:

$$(2.8)\quad \frac{d\left(\frac{dW_t}{dW_{t-j}}\right)}{dj} \approx \ln(1 + r) [1 - U'(W_0^{t-j} - W_{t-j})] (1 + r)^j + U''(W_0^{t-j} - W_{t-j}) (1 + r)^j [U_{t-j} + r (W_0^{t-j-1} - W_{t-j-1})]$$

The first term measures the effect of the passage of time on $-\frac{dW_T}{dU_{T-j}}$, weighing the marginal utility of memories. The second one, the effect over marginal utility of the changes in past memories.

It will be larger than zero when $U'(.) < 1$ iff

$$(2.9)\quad - [U_{t-j} + r (W_0^{t-j-1} - W_{t-j-1})] U''(W_0^{t-j} - W_{t-j}) / [1 - U'(W_0^{t-j} - W_{t-j})] < \ln(1 + r) \approx r$$

i.e., if $- \{U''(W_0^{t-j} - W_{t-j}) / [1 - U'(W_0^{t-j} - W_{t-j})]\} U_{t-j} < (- \{U''(W_0^{t-j} - W_{t-j}) / [1 - U'(W_0^{t-j} - W_{t-j})]\} (W_0^{t-j-1} - W_{t-j-1}) + 1) r$. If U(.) is convex, it will be…

It will be smaller than $\ln(1 + r) (1 + r)^j$ iff

$$(2.10)\quad \ln(1 + r) U'(W_0^{t-j} - W_{t-j}) \approx r [U'(W_0^{t-j} - W_{t-j}) - 1] > U''(W_0^{t-j} - W_{t-j}) [U_{t-j} + r (W_0^{t-j-1} - W_{t-j-1})]$$

or $$r [U'(W_0^{t-j} - W_{t-j}) - U''(W_0^{t-j} - W_{t-j}) (W_0^{t-j-1} - W_{t-j-1})] > U''(W_0^{t-j} - W_{t-j}) U_{t-j}$$

. If memories depreciate, we can postulate instead that:

$$(2.11)\qquad U_t = U[\sum_{i=1}^{t} U_i(1 - d)^{t-i}] = U(W'^{t}_0 - W'_t)$$

Eventually, $(1 - d)$ can be thought as $(1 - d) = (1 - d'') (1 + r)$, i.e., net of capitalization or net of compounding. Now, W'^{t}_0 differs from W_0^t as W'_t from W_t; W'_t and W'^{t}_0 are defined and linked according to:

$$(2.12)\quad W'^{T}_0 = W'_0 (1 - d)^T = \sum_{t=1}^{T} U_t (1 - d)^{T-t} + \sum_{t=1}^{\infty} \frac{U_{T+t}}{(1-d)^t} = \sum_{t=1}^{T} U_t (1 - d)^{T-t} + W'_T =$$

$$\text{(letting } j = T - t\text{:)}\qquad = \sum_{j=0}^{T-1} U_{T-j} (1 - d)^j + W'_T$$

W'_T appears as memory prospects at time T. $\frac{dW'_t}{dW'_{t-j}}\big|_{W_0}$ would have the same features of the previous ratio $\frac{dW_t}{dW_{t-j}}\big|_{W_0}$ with r replaced by – d:

$$(2.13) \qquad \frac{dW'_t}{dW'_{t-j}}\Big|_{W_0} = (1-d)^j$$

As long as d > 0, $\frac{dW'_t}{dW'_{t-j}}\big|_{W_0}$ decreases with j. Another interesting observation is that $W'^{t}_0 - W'_t$ may now decrease with t:

$$(2.14) \quad W'^{t}_0 - W'_t = \sum_{i=1}^{t} U_i(1-d)^{t-i} = (1-d)\sum_{i=1}^{t-1} U_i(1-d)^{t-1-i} + U_t = (1-d)(W'^{t-1}_0 - W'_{t-1}) + U_t$$

$W'^{t}_0 - W'_t > W'^{t-1}_0 - W'_{t-1}$ iff $U_t > d(W'^{t-1}_0 - W'_{t-1}) = d\sum_{i=1}^{t-1} U_i(1-d)^{t-1-i}$: the second argument increases in time iff d is small – if current felicity, U_t, is larger than depreciation of past memories.

We can now deduct that (1.7) still applies but now:

$$(2.15) \sum_{j=0}^{T-1} U_{T-j}(1-d)^j \Big/ \sum_{j=0}^{T-1-1} U_{T-1-j}(1-d)^j = (W'^{T}_0 - W'_T) / (W'^{T-1}_0 - W'_{T-1}) =$$

$$= (1-d) + U_T / (W'^{T-1}_0 - W'_{T-1}) = (1-d) + U_T \Big/ \Big[\sum_{j=0}^{T-1-1} U_{T-1-j}(1-d)^j\Big]$$

If U_t grows at a constant rate g, the expression will be equal to (1 – d) + (g + d) / {1 – [(1 - d) / (1 + g)]$^{T-1}$}. As T goes to ∞, it tends to 1 + g.

Welfare prospects at time T are given by:

$$(2.16) \qquad W_T = W_0^T - \sum_{j=0}^{T-1} U\Big[\sum_{i=1}^{T-j} U_i(1-d)^{T-j-i}\Big](1+r)^j$$

Then

$$(2.17) \qquad \frac{dW_t}{dW'_{t-j}} = U'(W'^{t-j}_0 - W'_{t-j})(1+r)^j$$

$$\frac{d\left(\frac{dW_t}{dW'_{t-j}}\right)}{dj} = \ln(1+r)\ U'(W'^{t-j}_0 - W'_{t-j})\ (1+r)^j + U''(W'^{t-j}_0 - W'_{t-j})\ (1+r)^j$$

$$\frac{d\left(W'^{t-j}_0 - W'_{t-j}\right)}{dj}$$

As $\frac{d\left(W'^{t-j}_0 - W'_{t-j}\right)}{dj} \approx \sum_{i=1}^{t-j-1} U_i\,(1-d)^{t-j-1-i} - \sum_{i=1}^{t-j} U_i\,(1-d)^{t-j-i} = -U_{t-j} + [\sum_{i=1}^{t-j-1} U_i\,(1-d)^{t-j-1-i}]\,(1-1+d) = -U_{t-j} + d\,[W'^{t-j-1}_0 - W'_{t-j-1}]$,

$$(2.18)\quad \frac{d\left(\frac{dW_t}{dW'_{t-j}}\right)}{dj} \approx \ln(1+r)\ U'(W'^{t-j}_0 - W'_{t-j})\ (1+r)^j - U''(W'^{t-j}_0 - W'_{t-j})\ (1+r)^j\ [U_{t-j} - d\,(W'^{t-j-1}_0 - W'_{t-j-1})]$$

This will be smaller than 0 iff:

$$(2.19)\quad \ln(1+r)\ U'(W'^{t-j}_0 - W'_{t-j}) \approx r\ U'(W'^{t-j}_0 - W'_{t-j}) < U''(W'^{t-j}_0 - W'_{t-j})\ [U_{t-j} - d\,(W'^{t-j-1}_0 - W'_{t-j-1})]$$

or

$$(2.20)\quad r < [U''(W'^{t-j}_0 - W'_{t-j}) / U'(W'^{t-j}_0 - W'_{t-j})]\ [U_{t-j} - d\,(W'^{t-j-1}_0 - W'_{t-j-1})]$$

The condition parallels (2.10). The interest rate is smaller than the symmetric of the Arrow-Pratt measure of risk aversion – a measure of convexity of the felicity function – times the utility net of depreciation of aggregate past welfare.

Proposition 2: 2.1: Under conventional time discounting and retrospect felicity functions, the individual accepts a trade-off of accumulated future prospects at a (future) moment in time with past ones at a rate given by (2.6); the rate may no longer be larger than 1 and especially it may decrease with the time distance.

2.2 Under conventional time discounting and retrospect felicity functions embedding fading-memory effects, the individual accepts a trade-off of accumulated future prospects at a (future) moment in time with past ones at a rate given by (2.17); again the rate may no longer be larger than 1 and especially it may decrease with the time distance.

3. Consumption Choices

. Let us now operationalize the concept in terms of decision-making. Let felicity depend on current consumption and past and current welfare so that $U_t = U[C_t, \sum_{i=1}^{t} U_i(1 - d)^{t-i}]$. The felicity function entails memories of past-to present welfare, fading with time at rate d per period. We also want to distinguish the market rate of interest from the individual's discount rate, and assign to him an initial wealth endowment, W_0. Capital markets are perfect and offer an interest rate (r) on deposited funds – to be used to consume and save in future time, constant and exogenous to the individual. Obviously, his budget constraint, that we expect he will exhaust, is:

$$(3.1) \qquad W_0 = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{1}{(1+r)^T} \sum_{t=1}^{\infty} \frac{C_{T+t}}{(1+r)^t}$$

Its mechanics in terms of C_t obey the rules exposed in section 1. A constant C_t, implies that (a perpetuity application of W_0):

$$(3.2) \qquad C_t = r\, W_0 , \qquad t = 1,2,...$$

If C_t is to grow at rate g < r: $W_0 = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t} = C_0 \sum_{t=1}^{\infty} \left(\frac{1+g}{1+r}\right)^t = C_0 \frac{\left(\frac{1+g}{1+r}\right)}{1-\left(\frac{1+g}{1+r}\right)} = \frac{C_1}{r-g}$; i.e., $C_1 = (r – g)\, W_0$,

$$(3.3) \qquad C_t = (r – g)\, W_0\, (1 + g)^{t-1} = \frac{r-g}{1+g} W_0\, (1 + g)^t, \qquad t = 1,2,...$$

Of course, for convergency of the series, g < r. g may even be negative, in which case C_t decreases over time – and therefore to 0, where the individual "lives" only of past memories - at rate –g; then, (for positive r) g + r > - 2…

. The individual's rational life-cycle path will maximize

$$(3.4) \qquad \sum_{t=1}^{\infty} U[C_t, \sum_{i=1}^{t} U_i(1 - d)^{t-i}]\, (1 + \rho)^{-t}$$

Current period felicity $U_t = U[C_t, \sum_{i=1}^{t} U_i(1 - d)^{t-i}]$. d = 1 renders the function to:

$$(3.5) \qquad U_t = U(C_t, U_t)$$

where echoes are still considered. $dU_t / dC_t = U_C(C_t, U_t) / [1 - U_R(C_t, U_t)]$. With further disappearance of echoes, $U_R(C_t, U_t) = 0$.

. (3.4) is maximized subject to (3.1). But it will be consistent, i.e., the agent knows that he will also solve similar problems every future period. At any point in time, he cannot change the past; but he can anticipate that his near future's actions may be more or less regrettable in the less near future… Then, at any time T:

$$(3.6) \qquad \underset{C_{T+1}, C_{T+2},\ldots}{Max} \sum_{t=1}^{\infty} U_{T+t} (1 + \rho)^{-t}$$

s.t.: $$(3.7) \qquad U_{T+t} = U[C_{T+t}, \sum_{i=1}^{T} U_i(1 - d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}], \ t = 1,2...$$

$$(3.8) \qquad W_T = W_0{}^T - \sum_{j=0}^{T-1} C_{T-j} (1 + r)^j = \sum_{t=1}^{\infty} C_{T+t} (1 + r)^{-t}$$

Given W_T and $U_1, U_2,\ldots,U_T$.

Or (as (3.7) holds also for t = -T + 1, - T + 2,…,0) $W_0{}^T$ ($W_0 (1 + r)^T$) and $C_1, C_2,\ldots,C_T$

The problem of time T gives a solution for $(C_{T+1}, C_{T+2}, \ldots)^T$ contingent on $(C_1, C_2, \ldots, C_T)$. Consistency is guaranteed iff all future solutions for any C_τ coincide and therefore coincide with that of T = 0 – that requires that FOC for any T give the same recurrent relation for any C_τ – and U_τ.

In Lagrangean form the problem becomes:

$$(3.9) \qquad \underset{\substack{C_{T+1}, C_{T+2},\ldots \\ U_{T+1}, U_{T+2},\ldots \\ \mu_T, \lambda_{T,1}, \lambda_{T,2},\ldots}}{Max} \sum_{t=1}^{\infty} U_{T+t} (1 + \rho)^{-t} - \sum_{t=1}^{\infty} \lambda_{T,t} \{U_{T+t} - U[C_{T+t}, \sum_{i=1}^{T} U_i (1 - d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}]\} + \mu_T [W_T - \sum_{t=1}^{\infty} C_{T+t} (1 + r)^{-t}]$$

FOC imply:

$$(3.10)\quad \frac{\partial W}{\partial U_{T+t}} = (1+\rho)^{-t} - \lambda_{T,t} + \sum_{j=0}^{\infty} \lambda_{T,t+j}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i\,(1-d)^{T+t+j-i}]\ (1-d)^j = 0$$

$$(3.11)\quad \frac{\partial W}{\partial C_{T+t}} = \lambda_{T,t}\, U_C[C_{T+t}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i\,(1-d)^{T+t-i}] - \mu_T\,(1+r)^{-t} = 0$$

Transversality conditions are $\lim_{t\to\infty} \lambda_{T,t}\, U_{T+t} = 0$, and $\lim_{T\to\infty} \mu_T\, W_T = 0$ – replacing terminal conditions of a finite time horizon problem.

From the second FOC:

$$(3.12)\qquad \lambda_{T,t} = \mu_T\,(1+r)^{-t} / U_C[C_{T+t}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i\,(1-d)^{T+t-i}]$$

Replacing in the first one:

$$(3.13)\quad (1+\rho)^{-t} + \mu_T \sum_{j=0}^{\infty} \{(1+r)^{-t-j} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i\,(1-d)^{T+t+j-i}]\}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i\,(1-d)^{T+t+j-i}]\ (1-d)^j = \mu_T\,(1+r)^{-t} / U_C[C_{T+t}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i\,(1-d)^{T+t-i}]$$

Then:

$$(3.14)\quad [(1+\rho)/(1+r)]^{-t} = \mu_T\,\{1 / U_C[C_{T+t}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i\,(1-d)^{T+t-i}] - \sum_{j=0}^{\infty} \{(1+r)^{-j} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i\,(1-d)^{T+t+j-i}]\}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i\,(1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i\,(1-d)^{T+t+j-i}]\ (1-d)^j\,\} \quad , \quad t = 1,2,\ldots,n$$

(3.14) will also be valid for t + 1; dividing the equation for t + 1 by that of t:

(3.15) $(1 + r) / (1 + \rho) = \{1 / U_C[C_{T+t+1}, \sum_{i=1}^{T} U_i (1 - d)^{T+t+1-i} + \sum_{i=T+1}^{T+t+1} U_i (1 - d)^{T+t+1-i}] - \sum_{j=0}^{\infty} \{(1 + r)^{-j} / U_C[C_{T+t+1+j}, \sum_{i=1}^{T} U_i (1 - d)^{T+t+j+1-i} + \sum_{i=T+1}^{T+t+j+1} U_i (1 - d)^{T+t+j+1-i}]\} U_R[C_{T+t+1+j}, \sum_{i=1}^{T} U_i (1 - d)^{T+t+1+j-i} + \sum_{i=T+1}^{T+t+1+j} U_i (1 - d)^{T+t+1+j-i}] (1 - d)^j \} /$

$/ \{1 / U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1 - d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}] - \sum_{j=0}^{\infty} \{(1 + r)^{-j} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i (1 - d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1 - d)^{T+t+j-i}]\} U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i (1 - d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1 - d)^{T+t+j-i}] (1 - d)^j \}$, $t = 1,2,...$

More compactly:

(3.16) $(1 + r) / (1 + \rho) = \{1 / U_C(C_{T+t+1}, W'^{T+t+1}_0 – W'_{T+t+1}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j}) / U_C(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j})] [(1 – d) / (1 + r)]^j \} /$

$/ \{1 / U_C(C_{T+t}, W'^{T+t}_0 – W'_{T+t}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+j}, W'^{T+t+j}_0 – W'_{T+t+j}) / U_C(C_{T+t+j}, W'^{T+t+j}_0 – W'_{T+t+j})] [(1 – d) / (1 + r)]^j \} =$

$= \{1 / U_C(C_{T+t+1}, W'^{T+t+1}_0 – W'_{T+t+1}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j}) / U_C(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j})] [(1 – d) / (1 + r)]^j \} /$

$/ \{1 / U_C(C_{T+t}, W'^{T+t}_0 – W'_{T+t}) - [U_R(C_{T+t}, W'^{T+t}_0 – W'_{T+t}) / U_C(C_{T+t}, W'^{T+t}_0 – W'_{T+t})] - [(1 – d) / (1 + r)] \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'^{T+t+j+1}_0 – W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'^{T+t+j+1}_0 – W'_{T+t+j+1})] [(1 – d) / (1 + r)]^j \}$, $t = 1,2,...$

$1 / U_C(C_{T+t+1}, W'^{T+t+1}_0 – W'_{T+t+1}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j}) / U_C(C_{T+t+1+j}, W'^{T+t+1+j}_0 – W'_{T+t+1+j})] [(1 – d) / (1 + r)]^j =$

$= [(1 + r) / (1 + \rho)] \{1 / U_C(C_{T+t}, W'^{T+t}_0 – W'_{T+t}) - [U_R(C_{T+t}, W'^{T+t}_0 – W'_{T+t}) / U_C(C_{T+t}, W'^{T+t}_0 – W'_{T+t})] - [(1 – d) / (1 + r)] \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'^{T+t+j+1}_0 – W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'^{T+t+j+1}_0 – W'_{T+t+j+1})] [(1 – d) / (1 + r)]^j \}$

Or

(3.17) $[(1+\rho)/(\rho+d)]\ \{1/U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) - [(1+r)/(1+\rho)] / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) + [(1+r)/(1+\rho)]\ U_R(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t})\} = [(1+\rho)/(\rho+d)]\ [1/U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t})]$

$\{U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) - [(1+r)/(1+\rho)]\ [1 - U_R(C_{T+t}, W'_0{}^{T+t} - W'_{T+t})]\} =$

$= \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'_0{}^{T+t+j+1} - W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'_0{}^{T+t+j+1} - W'_{T+t+j+1})]\ [(1-d)/(1+r)]^j \qquad t = 1,2,...,3$

The first interesting conclusion is that, in spite of complexity, consistency appears to be warranted [5]: (3.17) holds for any $T + t = \tau$: provided every generation optimizes, including $T = 0$, at any point T, the previous utilities that cannot be changed any longer and are arguments of the problem T, are – were - always the optimal ones (chosen at 0, 1, …, T-1) and do not colide with future upgrading.

Also, as

(3.18) $\sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'_0{}^{T+t+j+1} - W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'_0{}^{T+t+j+1} - W'_{T+t+j+1})]\ [(1-d)/(1+r)]^j = [(1-d)/(1+r)] \sum_{j=0}^{\infty} [U_R(C_{T+t+j+2}, W'_0{}^{T+t+j+2} - W'_{T+t+j+2}) / U_C(C_{T+t+j+2}, W'_0{}^{T+t+j+2} - W'_{T+t+j+2})]\ [(1-d)/(1+r)]^j + U_R(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1})]$

we can replace in (3.17) and use the lead left hand-side to conclude:

$[(1+\rho)/(\rho+d)]\ \{1/U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) - [(1+r)/(1+\rho)] / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) + [(1+r)/(1+\rho)]\ U_R(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t})\} = U_R(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1})] + [(1-d)/(1+r)]\ [(1+\rho)/(\rho+d)]\ \{1/U_C(C_{T+t+2}, W'_0{}^{T+t+2} - W'_{T+t+2}) - [(1+r)/(1+\rho)] / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) + [(1+r)/(1+\rho)]\ U_R(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1})\}$

or

(3.19) $\{1/U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) - [(1+r)/(1+\rho)] / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) + [(1+r)/(1+\rho)]\ U_R(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t})\} =$

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[5] Addiction, in habit formation models, can also be compatible with rational and time consistent behavior. See Becker and Murphy (1988) for example.

$= [(1 - d) / (1 + r)] \ \{1 / U_C(C_{T+t+2}, W'^{T+t+2}_0 - W'_{T+t+2}) - [(1 + r) / (1 + \rho)] / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) + [(1 + r) / (1 + \rho)] U_R(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1})\} + [(\rho + d) / (1 + \rho)] U_R(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1})] =$

$= [(1 - d) / (1 + r)] \ \{1 / U_C(C_{T+t+2}, W'^{T+t+2}_0 - W'_{T+t+2}) - [(1 + r) / (1 + \rho)] / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) \} + U_R(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1})]$

. We can then choose T = 0 for a full solution and recognize:

(3.20) $1 / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) - [(1 + r) / (1 + \rho)] / U_C(C_t, W'^t_0 - W'_t) + [(1 + r) / (1 + \rho)] U_R(C_t, W'^t_0 - W'_t) / U_C(C_t, W'^t_0 - W'_t) =$

$= [(1 - d) / (1 + r)] \ \{1 / U_C(C_{t+2}, W'^{t+2}_0 - W'_{t+2}) - [(1 + r) / (1 + \rho)] / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) \} + U_R(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1})]$

or

(3.21) $U_C(C_t, W'^t_0 - W'_t) / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] - [(1 - d) / (1 + r)] \ U_C(C_t, W'^t_0 - W'_t) / U_C(C_{t+2}, W'^{t+2}_0 - W'_{t+2}) - [(1 + r) / (1 + \rho)] =$

$= U_R(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) U_C(C_t, W'^t_0 - W'_t) / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1}) - [(1 + r) / (1 + \rho)] U_R(C_t, W'^t_0 - W'_t)$

(3.21) suggests that a balanced path for consumption is possible with a - steady-state - constant $U_C(C_t, W'^t_0 - W'_t) / U_C(C_{t+1}, W'^{t+1}_0 - W'_{t+1})$ and constant $U_R(C_t, W'^t_0 - W'_t)$. Then, provided d is small (or negative) so that the second argument increases with time, C_t should be increasing[6] if $U_{RC}(C_t, W'^t_0 - W'_t) > 0$[7] (provided that $U_{RR}(C_t, W'^t_0 - W'_t) < 0$) that is, the two arguments are complements. It will be decreasing if $U_{RC}(C_t, W'^t_0 - W'_t) < 0$ – and the arguments are substitutes. A large d may reverse the implications.

If d = 1 and $U_t = U(C_t, U_t)$:

(3.22) $U_C(C_t, U_t) / U_C(C_{t+1}, U_{t+1}) [1 - U_R(C_{t+1}, U_{t+1})] = (1 + r) / (1 + \rho) [1 - U_R(C_t, U_t)]$

If additionally we consider $U_R(C_{t+1}, U_{t+1})$ = constant – for instance, equal to zero -, we have the standard case:

[6] Habit, *addiction to consumption*, C_t, would be mimicked… Even if here we only have one good.

[7] We would have that consumption and past welfare enjoy complementarity in the sense of Orphanides and Zervos (1995).

(3.23) $U_C(C_t, U_t) / U_C(C_{t+1}, U_{t+1}) = (1 + r) / (1 + \rho)$

. Let d be free and consider that U_R tend to 0 – say, a possible steady-state -, the left hand-side is of (3.21) is equated to zero. Then:

(3.24) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] - [(1 - d) / (1 + r)] U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2}) = [(1 + r) / (1 + \rho)]$

or

$U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) \{1 + (1 - d) / (1 + \rho) - [(1 - d) / (1 + r)] U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2})\} = (1 + r) / (1 + \rho)$

Then another steady-state value of $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ would appear to occur. If along the way a long-run solution with intertemporal MRS constant:

$- [(1 - d) / (1 + r)] [U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2})]^2 +$
$+ U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] -$
$- (1 + r) / (1 + \rho) = 0$

Then, one can solve for the steady-state value of:

(3.25) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) = U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2}) = ([1 + (1 - d) / (1 + \rho)] \pm \{[1 + (1 - d) / (1 + \rho)]^2 - 4 (1 - d) / (1 + \rho)]\}^{1/2}) [(1 + r) / (1 - d)] / 2 = ([1 + (1 - d) / (1 + \rho)] \pm \{[1 - (1 - d) / (1 + \rho)]^2]\}^{1/2}) [(1 + r) / (1 - d)] / 2 = \{[1 + (1 - d) / (1 + \rho)] \pm [1 - (1 - d) / (1 + \rho)]\} [(1 + r) / (1 - d)] / 2$

Then, in such a steady-state, $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ equals (with the plus sign) $(1 + r) / (1 - d) > 1$; or (with the minus sign) $(1 + r) / (1 + \rho)$ (the value of the standard model).

A constant $C = C_t = r W_0$ will be then possible. If $U_{CR} > 0$ and retrospective wealth increases, $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) < 1$ - only possible with the negative root.

C_t may decrease over time; as $U_{CC} < 0$, $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) < 1$ if $U_{CR} > 0$ and retrospective wealth increases – again, only possible with the negative root.

If C_t increases over time as $U_{CC} < 0$, $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) > 1$ if $U_{CR} < 0$ and the sign of the root can be positive or negative; if $U_{CR} > 0$, we may achieve a constant $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$.

Proposition 3: Under conventional time discounting and retrospect felicity functions embedding eventual fading memory effects,

3.1 standard optimization is time consistent – allowing conclusions to be drawn from the beginning moment dated problem.

3.2 if d is small, an allocation of wealth with perfect capital markets and constant market interest rate, may have a steady-state over (3.21), provided a constant $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ and a constant $U_R(C_t, W'_0{}^t - W'_t)$ are compatible. If C_t and memories are complements - $U_{RC}(C_t, W'_0{}^t - W'_t) > 0$ -, and $U_{RR}(C_t, W'_0{}^t - W'_t) < 0$, consumption will increase along the steady-state – or rather optimal balanced - path.

4. Capital Accumulation

. Suppose we stage the new intertemporal preferences on the Ramsey's growth model. The representative household problem becomes:

$$(4.1) \quad \underset{C_{T+1}, C_{T+2}, \ldots}{Max} \sum_{t=1}^{\infty} U[C_{T+t}, \sum_{i=1}^{T} U_i (1 - d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}] (1 + \rho)^{-t}$$

$$\text{s.t. } (4.2) \quad U_{T+t} = U[C_{T+t}, \sum_{i=1}^{T} U_i(1 - d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}], \ t = 1,2...$$

$$(4.3) \quad (1 + n) k_{T+t} = (1 - d') k_{T+t-1} + f(k_{T+t-1}) - C_{T+t}$$

Given k_T and $U_1, U_2,\ldots,U_T$.

Or (as (4.2) holds also for t = -T + 1, - T + 2,…,0) k_T and $C_1, C_2,\ldots,C_T$

which in Lagrangean form becomes:

$$(4.4) \quad \underset{\substack{C_{T+1}, C_{T+2}, \ldots \\ k_{T+1}, k_{T+2}, \ldots \\ U_{T+1}, U_{T+2}, \ldots \\ \lambda_{T,1}, \lambda_{T,2}, \ldots \\ \mu_{T+1}, \mu_{T+2}, \ldots}}{Max} \sum_{t=1}^{\infty} U_{T+t} (1 + \rho)^{-t} - \sum_{t=1}^{\infty} \lambda_{T,t} \{U_{T+t} - U[C_{T+t}, \sum_{i=1}^{T} U_i (1 - d)^{T+t-i} +$$

$$\sum_{i=T+1}^{T+t} U_i (1 - d)^{T+t-i}]\} + \sum_{t=1}^{\infty} \mu_{T+t} [(1 - d') k_{T+t-1} + f(k_{T+t-1}) - C_{T+t} - (1 + n) k_{T+t}]$$

Transversality conditions are $\lim_{t\to\infty} \lambda_{T,t} U_{T+t} = 0$, and $\lim_{t\to\infty} \mu_{T+t} k_{T+t} = 0$.

FOC imply:

(4.5) $\frac{\partial W}{\partial U_{T+t}} = (1+\rho)^{-t} - \lambda_{T,t} + \sum_{j=0}^{\infty} \lambda_{T,t+j}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\ (1-d)^j = 0$, t = 1,2,...,n

(4.6) $\frac{\partial W}{\partial C_{T+t}} = \lambda_{T,t}\, U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1-d)^{T+t-i}] - \mu_{T+t} = 0$, t = 1,2,...,n

(4.7) $\frac{\partial W}{\partial k_{T+t}} = -\ \mu_{T+t} (1+n) + \mu_{T+t+1} [(1-d') + f'(k_{T+t})] = 0$, t = 1,2,...,n

From the second one:

(4.8) $\lambda_{T,t} = \mu_{T+t} / U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1-d)^{T+t-i}]$, t = 1,2,...,n

Replacing in the first one:

(4.9) $(1+\rho)^{-t} + \sum_{j=0}^{\infty} \{\mu_{T+t+j} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\ (1-d)^j = \mu_{T+t} / U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1-d)^{T+t-i}]$

From the third FOC:

(4.10) $\mu_{T+t+1} / \mu_{T+t} = (1+n) / [(1-d') + f'(k_{T+t})]$

and

(4.11) $\mu_{T+t+j} = \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+j})]\}\ \mu_{T+t}$

Then:

(4.12) $(1+\rho)^{-t} = \mu_{T+t} \{1 / U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1-d)^{T+t-i}] - \sum_{j=0}^{\infty} \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+j})]\} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\}\ U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\ (1-d)^j \}$, t = 1,2,...,n

(4.12) will also be valid for t + 1; dividing the equation for t + 1 by that of t, and using the fact that, from the third order condition:

(4.13) $[(1-d') + f'(k_{T+t})] / [(1+n)(1+\rho)] =$

$\{1 / U_C[C_{T+t+1}, \sum_{i=1}^{T} U_i (1-d)^{T+t+1-i} + \sum_{i=T+1}^{T+t+1} U_i (1-d)^{T+t+1-i}] - \sum_{j=0}^{\infty} \{\prod_{i=0}^{j} (1+n) / [(1 - d') + f'(k_{T+t+1+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+1+j})]\} / U_C[C_{T+t+1+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j+1-i} + \sum_{i=T+1}^{T+t+j+1} U_i (1-d)^{T+t+j+1-i}]\} U_R[C_{T+t+1+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+1+j-i} + \sum_{i=T+1}^{T+t+1+j} U_i (1-d)^{T+t+1+j-i}] (1-d)^j$
$\}$ /

/ $\{1 / U_C[C_{T+t}, \sum_{i=1}^{T} U_i (1-d)^{T+t-i} + \sum_{i=T+1}^{T+t} U_i (1-d)^{T+t-i}] - \sum_{j=0}^{\infty} \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+j})]\} / U_C[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}]\} U_R[C_{T+t+j}, \sum_{i=1}^{T} U_i (1-d)^{T+t+j-i} + \sum_{i=T+1}^{T+t+j} U_i (1-d)^{T+t+j-i}] (1-d)^j \}$

More compactly:

(4.14) $[(1-d') + f'(k_{T+t})] / [(1+n)(1+\rho)] = \{1 / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+1+j}, W'^{T+t+1+j}_0 - W'_{T+t+1+j}) / U_C(C_{T+t+1+j}, W'^{T+t+1+j}_0 - W'_{T+t+1+j})] (1-d)^j \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+1+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+1+j})]\} \}$ /

/ $\{1 / U_C(C_{T+t}, W'^{T+t}_0 - W'_{T+t}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+j}, W'^{T+t+j}_0 - W'_{T+t+j}) / U_C(C_{T+t+j}, W'^{T+t+j}_0 - W'_{T+t+j})] (1-d)^j \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+j})]\} \} =$

$= \{1 / U_C(C_{T+t+1}, W'^{T+t+1}_0 - W'_{T+t+1}) - \sum_{j=0}^{\infty} [U_R(C_{T+t+1+j}, W'^{T+t+1+j}_0 - W'_{T+t+1+j}) / U_C(C_{T+t+1+j}, W'^{T+t+1+j}_0 - W'_{T+t+1+j})] (1-d)^j \{\prod_{i=0}^{j} (1+n) / [(1-d') + f'(k_{T+t+1+i})]\} / \{(1+n) / [(1-d') + f'(k_{T+t+1+j})]\} \}$ /

/ $\{1 / U_C(C_{T+t}, W'^{T+t}_0 - W'_{T+t}) - [U_R(C_{T+t}, W'^{T+t}_0 - W'_{T+t}) / U_C(C_{T+t}, W'^{T+t}_0 - W'_{T+t})] - \{(1-d)(1+n) / [(1-d') + f'(k_{T+t})]\} \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'^{T+t+j+1}_0 - W'_{T+t+j+1})$ /

$U_C(C_{T+t+j+1}, W'^{\,T+t+j+1}_0 - W'_{T+t+j+1})] \, (1 - d)^j \, \{\prod_{i=0}^{j} (1 + n) / [(1 - d') + f'(k_{T+t+1+i})]\} / \{(1 + n) / [(1 - d') + f'(k_{T+t+1+j})]\} \}$

Or

(4.15) $1 / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) - \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} / U_C(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) + \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} \, U_R(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) / U_C(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) = [(\rho + d) / (1 + \rho)] \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'^{\,T+t+j+1}_0 - W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'^{\,T+t+j+1}_0 - W'_{T+t+j+1})] \, (1 - d)^j \, \{\prod_{i=0}^{j} (1 + n) / [(1 - d') + f'(k_{T+t+1+i})]\} / \{(1 + n) / [(1 - d') + f'(k_{T+t+1+j})]\} =$

$= \{(1 - d)(1 + n) / [(1 - d') + f'(k_{T+t+1})]\} \, \{1 / U_C(C_{T+t+2}, W'^{\,T+t+2}_0 - W'_{T+t+2}) - \{[(1 - d') + f'(k_{T+t+1})] / [(1 + n)(1 + \rho)]\} / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) + \{[(1 - d') + f'(k_{T+t+1})] / [(1 + n)(1 + \rho)]\} \, U_R(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t-1}, W'^{\,T+t+1}_0 - W'_{T+t+1})\} + [(\rho + d) / (1 + \rho)] \, [U_R(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1})]$

using the fact that:

$[(1 + \rho) / [(\rho + d)] \, \{1 / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) + \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} \, U_R(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) / U_C(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) - \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} / U_C(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) \} =$

$= \sum_{j=0}^{\infty} [U_R(C_{T+t+j+1}, W'^{\,T+t+j+1}_0 - W'_{T+t+j+1}) / U_C(C_{T+t+j+1}, W'^{\,T+t+j+1}_0 - W'_{T+t+j+1})] \, (1 - d)^j \, \{\prod_{i=0}^{j} (1 + n) / [(1 - d') + f'(k_{T+t+1+i})]\} / \{(1 + n) / [(1 - d') + f'(k_{T+t+1+j})]\}$

$= [U_R(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1})] +$

$\{(1 - d)(1 + n) / [(1 - d') + f'(k_{T+t+2})]\} \sum_{j=0}^{\infty} [U_R(C_{T+t+j+2}, W'^{\,T+t+j+2}_0 - W'_{T+t+j+2}) / U_C(C_{T+t+j+2}, W'^{\,T+t+j+2}_0 - W'_{T+t+j+2})] \, (1 - d)^j \, \{\prod_{i=0}^{j} (1 + n) / [(1 - d') + f'(k_{T+t+1+i})]\} / \{(1 + n) / [(1 - d') + f'(k_{T+t+2+j})]\}$

(4.15) $U_C(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) / U_C(C_{T+t+1}, W'^{\,T+t+1}_0 - W'_{T+t+1}) - \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} + \{[(1 - d') + f'(k_{T+t})] / [(1 + n)(1 + \rho)]\} \, U_R(C_{T+t}, W'^{\,T+t}_0 - W'_{T+t}) =$

$= \{(1 - d)(1 + n) / [(1 - d') + f'(k_{T+t+1})]\} \; \{ U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t+2}, W'_0{}^{T+t+2} - W'_{T+t+2}) - \{[(1 - d') + f'(k_{T+t+1})] / [(1 + n)(1 + \rho)]\} U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) + \{[(1 - d') + f'(k_{T+t+1})] / [(1 + n)(1 + \rho)]\} U_R(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t-1}, W'_0{}^{T+t+1} - W'_{T+t+1}) \} + [(\rho + d) / (1 + \rho)] [U_R(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1}) U_C(C_{T+t}, W'_0{}^{T+t} - W'_{T+t}) / U_C(C_{T+t+1}, W'_0{}^{T+t+1} - W'_{T+t+1})]$

Then consistency appears to hold; and then for T = 0:

(4.16) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] - \{(1 - d)(1 + n) / [(1 - d') + f'(k_{t+2})]\} U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2}) - [(1 - d') + f'(k_t)] / [(1 + n)(1 + \rho)] =$

$= U_R(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) - \{[(1 - d') + f'(k_{t+1})] / [(1 + n)(1 + \rho)]\} U_R(C_t, W'_0{}^t - W'_t)$

(4.16) seems analogous to (3.21).

Let d = 1. Then:

(4.17) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) - [(1 - d') + f'(k_t)] / [(1 + n)(1 + \rho)] =$

$= U_R(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) - \{[(1 - d') + f'(k_{t+1})] / [(1 + n)(1 + \rho)]\} U_R(C_t, W'_0{}^t - W'_t)$

With $U_R = 0$, we have the the standard Ramsey model solution:

(4.18) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) = [(1 - d') + f'(k_t)] / [(1 + n)(1 + \rho)]\}$

(4.18) is analogous to (3.23).
Let, as before, U_R tend to 0 after (4.16):

(4.19) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] - \{(1 - d)(1 + n) / [(1 - d') + f'(k_{t+2})]\} U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+2}, W'_0{}^{t+2} - W'_{t+2}) - [(1 - d') + f'(k_t)] / [(1 + n)(1 + \rho)] = 0$

With steady-state level of k* we can generate similar statements as those of section 3…

. Under (4.16), a steady-state solution for k* is possible. It may imply that $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ is constant and so is $U_R(C_t, W'_0{}^t - W'_t)$. Then:

Then the expression becomes – for constant k*:

(4.20) $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) [1 + (1 - d) / (1 + \rho)] - \{(1 - d) (1 + n) / [(1 - d') + f'(k^*)]\} [U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})]^2 -$
$- U_R(C_t, W'_0{}^t - W'_t) \{U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1}) - [(1 - d') + f'(k^*)] / [(1 + n) (1 + \rho)]\} = [(1 - d') + f'(k^*)] / [(1 + n) (1 + \rho)]$

Let us suppose that on such path U_t grows at rate g; then $W'_0{}^t - W'_t$ grows at rate $-d + (g + d) / \{1 - [(1 - d) / (1 + g)]^{t-1}\}$, that goes to g as t tends to ∞.

An additively separable felicity function in $W'_0{}^t - W'_t$ and C_t – implying $U_{RC}(C_t, W'_0{}^t - W'_t) = 0$ - for example, is compatible with a steady-state path with constant C – then $U_R(C_t, W'_0{}^t - W'_t)$ may tend to zero.

Homogeneous felicity functions in the two arguments may generate paths along which $U_R(C_t, W'_0{}^t - W'_t)$ is constant; g and $c = C_{t+1} / C_t - 1$, must then allow for $U_C(C_t, W'_0{}^t - W'_{T+t}) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ to remain constant as well.

If $U_R(C_t, W'_0{}^t - W'_t)$ tends to 0, one can expect that $U_C(C_t, W'_0{}^t - W'_t) / U_C(C_{t+1}, W'_0{}^{t+1} - W'_{t+1})$ will tend to $[(1 - d') + f'(k^*)] / [(1 + n) (1 + \rho)]$; being C constant, $[(1 + n) (1 + \rho)]$ divided by 1 plus the growth rate of marginal utility with respect to consumption will equal $[(1 - d') + f'(k^*)]$; the interest rate [8] will tend to be the same as in the Ramsey case; if $U_{RC} > 0$ – say, U(., .) is linear in $W'_0{}^t - W'_t$ -, it is possible that f'(k*) is now smaller than in that case and therefore, capital stock is larger.

Finally, a constant $U_R(C_t, W'_0{}^t - W'_t)$, if $U_{RR}(C_t, W'_0{}^t - W'_t) < 0$ but $U_{RC}(C_t, W'_0{}^t - W'_t) > 0$ [9], would allow for an increasing C_t [10] along a balanced path – while the stock of past welfare is increasing. $U_{RR}(C_t, W'_0{}^t - W'_t) > 0$, though, would be consistent with the opposite result. Clearly, consumption becomes more variable than without retrospective effects;

[8] The interest rate is still (as usual) the marginal rate of substitution – on a cumulative welfare indifference curve - between consumption of two consecutive periods minus 1 – but consumption enters future "past welfare" which should be accounted for in that MRS computation.

[9] We would have that consumption and past welfare enjoy complementarity in the sense of Orphanides and Zervos (1995).

[10] Habit, *addiction to consumption*, C_t, would be mimicked…

intuitively, that would be expected: now there is a stock that already accrues and provides a "weight" – a balanced source of felicity.

Proposition 4: Allowing retrospect felicity functions in the Ramsey growth model,

4.1 standard optimization allows for time consistency – allowing conclusions to be drawn from the beginning moment dated problem.

4.2 steady-states for capital are possible. Then if along the optimal balanced path $U_R(C_t, W'^{t}_{0} – W'_t)$ is constant (and $W'^{t}_{0} – W'_t$ increasing), if $U_{RR} < 0$ and $U_{CR} > 0$, consumption may be increasing. Steady-state capital may then be larger than in the standard Ramsey's world.

5. Applications: Vertical Systems

5.1. Vertical Systems and the Optimal Depth

. The principle has an obvious application in production theory – namely, in the modelling of vertical product chains.

Let Y_{T+t} denote the output of knot T + t in the production descending hierarchy – i.e., interpreted backwardly; each knot requires one unit of time to process. Maximum complexity is achieved at knot T = 0, where the final product Y_1 is decided or processed and requires G knots to develop. The problem faced – at any point in time - by all upstream knots to process T (T = 0,1,2,...,G-1; Y_{G+1} can be thought as raw material) is:

$$(5.1)\quad \underset{X_{T+1}, X_{T+2},\ldots,X_G, Y_{G+1}}{Max} \sum_{t=1}^{G-T} (P_{T+t}\, Y_{T+t} \;-\; W_{T+t}\, X_{T+t} \;-\; P_{T+t+1}\, Y_{T+t+1}) \prod_{i=1}^{t} (1 + \rho_{T+i}) = \sum_{t=1}^{\infty} \Pi_{T+t} \prod_{i=1}^{t} (1 + \rho_{T+i})$$

$$\text{s.t. } (5.2)\qquad Y_{T+t} = F^{T+t}[X_{T+t}, Y_{T+t+1}\,(1 - d)]\,,\quad t = 1,2\ldots, G - T$$

$$\text{Given } Y_1, Y_2,\ldots,Y_T$$

Or (as (5.2) holds also for t = -T + 1, - T + 2,…,0) $X_1, X_2,\ldots,X_T$

Each term of the objective function accounts for profits at tier T + t. The costs with the intermediate input - revenues of the immediately lower tier - were not discounted, which would impose an additional cost factor on the producer – an hypothesis that we proceed below. Optimal internal prices would be zero and the first term would disappear with the exception of $P_{T+1}\, Y_{T+1}$, and the last term in parenthesis would disapear with the exception of $P_{G+1}\, Y_{G+1}$.

The Lagrangean of the problem is then:

(5.3) $\underset{\substack{X_{T+1},X_{T+2},...,X_G \\ Y_{T+1},Y_{T+2},...,Y_G,Y_{G+1} \\ \lambda_{T+1},\lambda_{T+2},...,\lambda_G}}{Max}$ $P_{T+1}\ Y_{T+1}\ (1 + \rho_{T+1}) - \sum_{t=1}^{G-T} W_{T+t}\ X_{T+t} \prod_{i=1}^{t} (1 + \rho_{T+i}) - P_{G+1}$

$Y_{G+1} \prod_{i=1}^{G-T+1} (1 + \rho_{T+i}) + \sum_{t=1}^{G-T} \lambda_{T+t} \{- Y_{T+t} + F^{T+t}[X_{T+t}, Y_{T+t+1} (1 - d)]\}$

FOC yield, along with the restriction:

(5.4) $\frac{\partial W}{\partial Y_{T+1}} = P_{T+1} (1 + \rho_{T+1}) - \lambda_{T+1} = 0$

(5.5) $\frac{\partial W}{\partial Y_{T+t}} = - \lambda_{T+t} + \lambda_{T+t-1}\ F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t} (1 - d)]\ (1 - d) = 0,$ t = 2,3…, G - T

(5.6) $\frac{\partial W}{\partial X_{T+t}} = - \prod_{i=1}^{t} (1 + \rho_{T+i})\ W_{T+t} + \lambda_{T+t}\ F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)] = 0,$ t = 1,2,…, G - T

(5.7) $\frac{\partial W}{\partial Y_{G+1}} = P_{G+1} \prod_{i=1}^{G-T+1} (1 + \rho_{T+i}) - \lambda_G\ F^G{}_Y[X_G, Y_{G+1} (1 - d)]\ (1 – d) = 0$

From (5.6), Then the current value – at T - of the shadow price of Y_{T+t} is:

(5.8) $\lambda_{T+t} = \prod_{i=1}^{t} (1 + \rho_{T+i})\ W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)]$, t = 1,2,…, G - T

(5.8) is the shadow-price of Y_{T+t} in the supply chain at date T. If Y_{T+t} were available in the market - at time T + t – at price

(5.9) $P_{T+t} < \lambda_{T+t} / \prod_{i=1}^{t} (1 + \rho_{T+i}) = W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)],$

it would be worthwhile to buy Y_{T+t} rather than producing it. If $P_{T+t} > \lambda_{T+t} / \prod_{i=1}^{t} (1 + \rho_{T+i})$, it would be worthwhile to produce in excess of Y_{T+t}, the requirement of next knot.

Replacing in (5.4) and (5.5):

(5.10) $P_{T+1} = W_{T+1} / F^{T+1}{}_X[X_{T+1}, Y_{T+2} (1 - d)]$

and

(5.11) $(1 + \rho_{T+t})\ W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)] = W_{T+t-1} / F^{T+t-1}{}_X[X_{T+t-1}, Y_{T+t} (1 - d)]\ F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t} (1 - d)]\ (1 - d)$, t = 2,3…, G - T and

(5.12) $P_{G+1}(1+\rho_{G+1}) = F^{G}{}_{Y}[X_G, Y_{G+1}(1-d)](1-d)\, W_G / F^{G}{}_{X}[X_G, Y_{G+1}(1-d)]$

Consistency is no longer guaranteed, *if* Y_{T+1} is decided at level T – instead of level T – 1 or lower -, i.e., obeying (5.4) and (5.10); then, a unique form $\frac{\partial W}{\partial Y_{T+t}}$ does not hold for any T + t = τ. This provides a rational for vertical integration – or suggest quantity-fixing seeking by the uptream entity of the immediate downstream tier output under a decentralized vertical system…

The objection is withdrawn if (5.4) – and (5.7) only holds for Y_1 – for t = 1 at problem T = 0; or rather for Y_{T*+1} where T* is the optimal depth.

The optimal depth T* would be the one at which the most recent optimal profit – the marginal effect of switching from the problem of complexity (ordered descendingly) T + 1 to that of T (provided they increase with T) - becomes zero: $\Pi_{T*+1} - \Pi_{T*} = 0$, where Π_T denotes optimal overall optimal profits of the T – G-length supply chain.

$F^j(.,.)$ may or may not be knot-dependent – i.e., of j. If $F^j[X_j, Y_{j+1}(1-d)] = F[X_j, Y_{j+1}(1-d)]$, all j, and F(., .) results in just increased production, the problem solves for the optimal time-span and total Y_{T*+1}.

. An interesting special case is the one for which $\rho_j = 0$, all j. It would apply if all tiers work simultaneously. Then

$$(5.13) \quad \underset{\substack{X_{T+1},X_{T+2},\dots,X_G \\ Y_{T+1},Y_{T+2},\dots,Y_G,Y_{G+1} \\ \lambda_{T+1},\lambda_{T+2},\dots,\lambda_G}}{Max} \quad P_{T+1}Y_{T+1} - \sum_{t=1}^{G-T} W_{T+t}X_{T+t} - P_{G+1}Y_{G+1}$$

$$\text{s.t. } (5.14) \qquad Y_{T+t} = F^{T+t}[X_{T+t}, Y_{T+t+1}(1-d)], \quad t = 1,2\dots, G-T$$

$$\text{Given } Y_1, Y_2,\dots,Y_T$$

Then, optimality requires:

$$(5.15) \qquad P_{T+1} = W_{T+1} / F^{T+1}{}_{X}[X_{T+1}, Y_{T+2}(1-d)]$$

and

$$(5.16) \quad W_{T+t} / F^{T+t}{}_{X}[X_{T+t}, Y_{T+t+1}(1-d)] = W_{T+t-1} / F^{T+t-1}{}_{X}[X_{T+t-1}, Y_{T+t}(1-d)]\, F^{T+t-1}{}_{Y}[X_{T+t-1}, Y_{T+t}(1-d)]\,(1-d), \quad t = 2,3\dots, G-T$$

and

$$(5.17) \quad P_{G+1} = F^{G}{}_{Y}[X_G, Y_{G+1}(1-d)](1-d)\, W_G / F^{G}{}_{X}[X_G, Y_{G+1}(1-d)]$$

If Y_j is homogeneous (then P_j is constant for all j, possibly all j > G + 1), economies of depth at level t + T may be linked to $F^{T+t}[X_{T+t}, Y_{T+t+1}(1-d)] - Y_{T+t+1}(1-d) > F^{T+t}(X_{T+t}, 0)$.

. Another interesting modification considers that an integrated vertical system is being evaluated and factors must be paid in advance each period. Then the typical firm solves:

$$(5.18)\quad \underset{X_{T+1},X_{T+2},\dots,X_G,Y_{G+1}}{Max} \sum_{t=1}^{G-T} [P_{T+t}\ Y_{T+t}\ - (1+\rho_{T+1+t})\ W_{T+t}\ X_{T+t}\ - (1+\rho_{T+1+t})\ P_{T+t+1}\ Y_{T+t+1}] \prod_{i=1}^{t} (1+\rho_{T+i}) = \sum_{t=1}^{G-T} \Pi_{T+t} \prod_{i=1}^{t} (1+\rho_{T+i}) = (1+\rho_{T+1})\ P_{T+1}\ Y_{T+1} - \sum_{t=1}^{G-T} \prod_{i=1}^{t+1} (1+\rho_{T+i})\ W_{T+t}\ X_{T+t} - \prod_{i=1}^{G-T+1} (1+\rho_{T+i})\ P_{G+1}\ Y_{G+1}$$

s.t. (5.19) $\quad Y_{T+t} = F^{T+t}[X_{T+t}, Y_{T+t+1}(1-d)]$, $\quad t = 1,2..., T - G$

Given $Y_1, Y_2,\dots,Y_T$

The Lagrangean form becomes:

$$(5.20)\quad \underset{\substack{X_{T+1},X_{T+2},\dots,X_G\\ Y_{T+1},Y_{T+2},\dots,Y_G,Y_{G+1}\\ \lambda_{T+1},\lambda_{T+2},\dots,\lambda_G}}{Max} (1+\rho_{T+1})\ P_{T+1}\ Y_{T+1} - \sum_{t=1}^{G-T} \prod_{i=1}^{t+1} (1+\rho_{T+i})\ W_{T+t}\ X_{T+t} - \prod_{i=1}^{G-T+1} (1+\rho_{T+i})\ P_{G+1}\ Y_{G+1} + \sum_{t=1}^{G-T} \lambda_{T+t} \{-Y_{T+t} + F^{T+t}[X_{T+t}, Y_{T+t+1}(1-d)]\}$$

Then:

$$(5.21)\quad \frac{\partial W}{\partial Y_{T+1}} = P_{T+1}(1+\rho_{T+1}) - \lambda_{T+1} = 0$$

$$(5.22)\quad \frac{\partial W}{\partial Y_{T+t}} = -\lambda_{T+t} + \lambda_{T+t-1}\ F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t}(1-d)]\ (1-d) = 0,\quad t = 2,3,\dots,G\text{-}T$$

$$(5.23)\quad \frac{\partial W}{\partial X_{T+t}} = -\prod_{i=1}^{t+1} (1+\rho_{T+i})\ W_{T+t} + \lambda_{T+t}\ F^{T+t}{}_X[X_{T+t}, Y_{T+t+1}(1-d)] = 0,\quad t = 1,2,\dots,G\text{-}T$$

$$(5.24)\quad \frac{\partial W}{\partial Y_{G+1}} = -\prod_{i=1}^{G-T+1} (1+\rho_{T+i})\ P_{G+1} + \lambda_G\ F^G{}_Y[X_G, Y_{G+1}(1-d)]\ (1-d) = 0$$

Then the current value – at T - of the shadow price of Y_{T+t} is:

$$(5.25)\quad \lambda_{T+t} = \prod_{i=1}^{t+1} (1+\rho_{T+i})\ W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1}(1-d)]\ ,\ t = 1,2,\dots,G\text{-}T$$

Replacing:

$$(5.26)\quad P_{T+1} = (1+\rho_{T+2})\ W_{T+1} / F^{T+1}{}_X[X_{T+1}, Y_{T+2}(1-d)]$$

and

(5.27) $(1 + \rho_{T+t+1}) W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)] =$

$= (1 - d) W_{T+t-1} F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t} (1 - d)] / F^{T+t-1}{}_X[X_{T+t-1}, Y_{T+t} (1 - d)], \quad t = 2,3,\ldots, G-T$

and

(5.28) $P_{G+1} = F^G{}_Y[X_G, Y_{G+1} (1 - d)] (1 - d) W_G / F^G{}_X[X_G, Y_{G+1} (1 - d)]$

The two first conditions now allow for two different interpretations:

If the sequence of functions $F^j(.,)$ is the only technology available in the market, desintegrated supply chains can be efficient with prices obeying (5.25). Then, it must be the case – because consistency must then hold – that for any T

(5.29) $P_{T+t} = (1 + \rho_{T+t+1}) W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)] =$

$= P_{T+t-1} (1 - d) F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t} (1 - d)] / (1 + \rho_{T+t}) \;, \; t = 2,3,\ldots,G-T$

i.e., for any knot j:

(5.30) $P_j = (1 + \rho_{j+1}) W_j / F^j{}_X[X_j, Y_{j+1} (1 - d)] =$

$= P_{j-1} (1 - d) F^{j-1}{}_Y[X_{j-1}, Y_j (1 - d)] / (1 + \rho_j) \;, \; j = 2,3,\ldots,G$

If not, and P_{T+t} is the price at which Y_{T+t} is offered in the market, tier T+t-1 will be integrated with T+t provided that …

(5.31) $P_{T+t} < (1 + \rho_{T+t+1}) W_{T+t} / F^{T+t}{}_X[X_{T+t}, Y_{T+t+1} (1 - d)] = (1 - d) W_{T+t-1} F^{T+t-1}{}_Y[X_{T+t-1}, Y_{T+t} (1 - d)] / F^{T+t-1}{}_X[X_{T+t-1}, Y_{T+t} (1 - d)] < P_{T+t-1}$

Also, if X_{T+t-1} is not specific and must be uniformly priced at moment t, for observed vertical integration until j:

(5.32) $P_j F^j{}_X[X_j, Y_{j+1} (1 - d)] / (1 + \rho_{j+1}) = W_j$

or

(5.33) $P_j F^j{}_X[X_j, Y_{j+1} (1 - d)] / W_j = (1 + \rho_{j+1})$

As P_j decreases with j, and if $F^j{}_X[X_j, Y_{j+1} (1 - d)] = F_X[X_j, Y_{j+1} (1 - d)]$, $F_{XY}[X_j, Y_{j+1} (1 - d)] > 0$, and only depends on X, we would expect that – if W is constant - ρ_{j+1} to decrease with j – to decrease with the distance to the delivery as a final product. Then this implies a decreasing interest rate with the distance to maturity [11].

Alternatively:

[11] Some studies report a downward sloping term structure - see Shiller (1990), page 629.

(5.34) $(1 - d)\, F^{j-1}{}_{Y}[X_{j-1}, Y_j (1 - d)] = P_j / P_{j-1} \ (1 + \rho_{j+1})$

If the product is homogeneous and $P_j = P_{j-1}$, provided $F_{YY}[X_{j-1}, Y_j (1 - d)] < 0$, and $F_{YX}[X_{j-1}, Y_j (1 - d)]$ is negligible, ρ_{j+1} must increase with j. Conversely, interest rate rises with term – decreases with time to maturity [12]. If $F_{YY}[X_{j-1}, Y_j (1 - d)]$ is negligible, and $F^{j-1}{}_{YX}[X_{j-1}, Y_j (1 - d)] > 0$, both term structures may emmerge. Note, however, that condition (5.34) is only observed for only integrated knots…

Proposition 5: Considering the product-chain problem of a conventional producer

5.1 standard optimization will not allow consistency unless the current tier product is decided at the downstream stage. Quantity-fixing practices may levy the ineficiency of eventual vertical desintegration.

5.2 (5.29) holds for "shadow" prices of a vertically integrated system.

5.3 A decreasing discount rate with time to maturity may be observed.

5.2. Vertical Systems in Growth Models:
5.2.1. Without Capital

. In a growth model, assuming there are T (ascending) hierarchic production levels. There are potential externalities across the production tiers so that, instead of $Y_{T-j,t} = F^{T-j}[X_{T-j,t}, Y_{T-j-1,t-1} (1 - d)]$, $Y_{T-j,t}$ represents value-added at tier T-j and obeys (6.2):

(6.1) $$\underset{X_{T+1}, X_{T+2}, \ldots, k_{T+1}, k_{T+2}, \ldots}{Max} \sum_{t=1}^{\infty} U(Y_{T,t})\,(1 + \rho)^{-t}$$

s.t.

(6.2) $$Y_{T-j,t} = F^{T-j}[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1 - d)^i] - \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1 - d)^i,\ j = 0,1...,T-1$$

(6.3) $$\sum_{j=0}^{T-1} X_{T-j,t} = X$$

Given $Y_{1,0}, Y_{2,0}, \ldots, Y_{T,0}$

The problem is equivalent to:

[12] See Shiller (1990), page 629.

(6.4) $\underset{\substack{X_{1,1},X_{1,2},\ldots,X_{T,1},X_{T,2},\ldots \\ Y_{1,1},Y_{1,2},\ldots,Y_{T,1},Y_{T,2},\ldots \\ \lambda_{1,1},\lambda_{1,2},\ldots,\lambda_{T,1},\lambda_{T,2},\ldots \\ \mu_1,\mu_2,\ldots \\ \eta_1,\eta_2,\ldots}}{Max} \sum_{t=1}^{\infty} U(Y_{T,t})\ (1+\rho)^{-t} + \sum_{t=1}^{\infty}\sum_{j=0}^{T-1} \lambda_{T-j,t}\ \{-\ Y_{T-j,t} + f^{T-j}[X_{T-j,t},$

$\sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}]\} + \sum_{t=1}^{\infty} \eta_t\,(X - \sum_{j=0}^{T-1} X_{T-j,t})$

Transversality conditions require that $\lim_{t\to\infty} \lambda_{T-j,t}\, Y_{T-j,t} = 0,\ j = 0,1,2,\ldots,T-1$

FOC yield:

(6.5) $\dfrac{\partial W}{\partial Y_{T,t}} = (1+\rho)^{-t}\, U'(Y_{T,t}) - \lambda_{T,t} = 0$

(6.6) $\dfrac{\partial W}{\partial X_{T-j,t}} = \lambda_{T-j,t}\, f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] - \eta_t = 0,\ j = 0,1,\ldots, T-1$

(6.7) $\dfrac{\partial W}{\partial Y_{T-j,t}} = -\ \lambda_{T-j,t} + \sum_{s=1}^{j} \lambda_{T-j+s,t+s}\ f^{T-j+s}{}_Y[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]$
$(1-d)^{s} = 0,\ \ j = 1,2,\ldots, T-1$

From (6.5) and (6.6)

(6.8) $\eta_t = \lambda_{T,t}\, f^{T}{}_X[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-i,t-i}\,(1-d)^{i}] = (1+\rho)^{-t}\, U'(Y_{T,t})\, f^{T}{}_X[X_{T,t},$
$\sum_{i=1}^{T-1} Y_{T-i,t-i}\,(1-d)^{i}] = \lambda_{T-j,t}\, f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}],\ j = 0,1,\ldots, T-1$

From (6.6) and (6.7)

(6.9) $\eta_t = f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] \sum_{s=1}^{j} \eta_{t+s}\,(1-d)^{s}\ f^{T-j+s}{}_Y[X_{T-j+s,t+s},$
$\sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}] / f^{T-j+s}{}_X[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s-i}\,(1-d)^{i}],\ \ j = 1,2,\ldots, T-1$

Replacing (6.8):

(6.10) $(1+\rho)^{-t}\, U'(Y_{T,t})\, f^{T}{}_X[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-i,t-i}\,(1-d)^{i}] =$

$$= f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i] \sum_{s=1}^{j} \{(1+\rho)^{-(t+s)} U'(Y_{T,t+s}) f^T{}_X[X_{T,t+s}, \sum_{i=1}^{T-1} Y_{T-i,t+s-i} (1-d)^i]\} (1-d)^s f^{T-j+s}{}_Y[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i} (1-d)^i] / f^{T-j+s}{}_X[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s-i} (1-d)^i] \quad , \quad j = 1,2,\ldots, T-1$$

or

$$(6.11) \quad U'(Y_{T,t}) f^T{}_X[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-i,t-i} (1-d)^i] =$$

$$= f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i] \sum_{s=1}^{j} \{(1+\rho)^{-s} U'(Y_{T,t+s}) f^T{}_X[X_{T,t+s}, \sum_{i=1}^{T-1} Y_{T-i,t+s-i} (1-d)^i]\} (1-d)^s f^{T-j+s}{}_Y[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i} (1-d)^i] / f^{T-j+s}{}_X[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s-i} (1-d)^i] \quad , \quad j = 1,2,\ldots, T-1$$

A steady-state – with $Y_{T-j,t} = Y_{T-j}^*$, $X_{T-j,t} = X_{T-j}^*$ – implies that all multipliers decrease at rate ρ. (6.10) implies:

$$(6.12) \quad f^{T-j}{}_X[X_{T-j}^*, \sum_{i=1}^{T-j-1} Y_{T-j-i}^* (1-d)^i] = 1 / \sum_{s=1}^{j} (1+\rho)^{-s} (1-d)^s f^{T-j+s}{}_Y[X_{T-j+s}^*, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i}^* (1-d)^i] / f^{T-j+s}{}_X[X_{T-j+s}^*, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i}^* (1-d)^i] \quad , \quad j = 1,2,\ldots, T-1$$

That is, optimality requires the inverse of the marginal product with respect to X at a knot T-j to equal the discounted sum of those above it, each weighted by the corresponding (properly depreciated) marginal product with respect to the intermediate product.

Proposition 6: Introducing the product-chain system into a representative infinitely lived agent's problem, a steady-state may exist with constant per capita aggregates and obeys (6.12), (6.2) and (6.3). Consistency is obeyed.

5.2.2. Vertical Systems in Growth Models: With Capital Accumulation

. With capital used in all knots, with $Y_{T-j,t} = F^{T-j}[X_{T-j,t}, k_{T-j}, Y_{T-j-1,t-1} (1-d)]$, build along with the final output:

$$(6.13) \quad \underset{X_{T+1}, X_{T+2},\ldots,k_{T+1},k_{T+2},\ldots}{Max} \quad \sum_{t=1}^{\infty} U(C_t) (1+\rho)^{-t}$$

s.t.

(6.14) $Y_{T\text{-}j,t} = F^{T\text{-}j}[X_{T\text{-}j,t}, k_{T\text{-}j,t\text{-}1}, \sum_{i=1}^{T-j-1} Y_{T\text{-}j\text{-}i,t\text{-}i} (1 - d)^i] + \sum_{i=1}^{T-j-1} Y_{T\text{-}j\text{-}i,t\text{-}i} (1 - d)^i =$

$= f^{T\text{-}j}[X_{T\text{-}j,t}, k_{T\text{-}j,t\text{-}1}, \sum_{i=1}^{T-j-1} Y_{T\text{-}j\text{-}i,t\text{-}i} (1 - d)^i]$, $j = 0,1..., T - 1$

(6.15) $$\sum_{j=0}^{T-1} X_{T\text{-}j,t} = X$$

(6.16) $$(1 + n) \sum_{j=0}^{T-1} k_{T\text{-}j,t} = (1 – d') \sum_{j=0}^{T-1} k_{T\text{-}j,t\text{-}1} + Y_{T,t} - C_t$$

Given $k_{1,0}, k_{2,0},\dots,k_{T,0}$ and $Y_{1,0}, Y_{2,0},\dots,Y_{T\text{-}1,0}$

The problem is equivalent to:

(6.17) $$\underset{\substack{C_1,C_2,\dots \\ X_{1,1},X_{1,2},\dots,X_{T,1},X_{T,2},\dots \\ k_{1,1},k_{1,2},\dots,k_{T,1},k_{T,2},\dots \\ Y_{1,1},Y_{1,2},\dots,Y_{T,1},Y_{T,2},\dots \\ \lambda_{1,1},\lambda_{1,2},\dots,\lambda_{T,1},\lambda_{T,2},\dots \\ \mu_1,\mu_2,\dots \\ \eta_1,\eta_2,\dots}}{Max} \sum_{t=1}^{\infty} U(C_t) (1 + \rho)^{-t} + \sum_{t=1}^{\infty} \sum_{j=0}^{T-1} \lambda_{T\text{-}j,t} \{- Y_{T\text{-}j,t} + f^{T\text{-}j}[X_{T\text{-}j,t}, k_{T\text{-}j,t\text{-}1}, \sum_{i=1}^{T-j-1} Y_{T\text{-}j\text{-}i,t\text{-}i} (1 - d)^i]\} + \sum_{t=1}^{\infty} \eta_t (X - \sum_{j=0}^{T-1} X_{T\text{-}j,t}) + \sum_{t=1}^{\infty} \mu_t [(1 – d') \sum_{j=0}^{T-1} k_{T\text{-}j,t\text{-}1} + Y_{T,t} - C_t - (1 + n) \sum_{j=0}^{T-1} k_{T\text{-}j,t}]$$

Transversality conditions require that $\lim_{t\to\infty} \lambda_{T\text{-}j,t} Y_{T\text{-}j,t} = 0$, $j = 1,2,\dots,T\text{-}1$, and $\lim_{t\to\infty} \mu_t \sum_{j=0}^{T-1} k_{T\text{-}j,t} = 0$.

FOC yield:

(6.18) $$\frac{\partial W}{\partial C_t} = (1 + \rho)^{-t} U_C(C_t) - \mu_t = 0$$

(6.19) $$\frac{\partial W}{\partial X_{T-j,t}} = - \eta_t + \lambda_{T\text{-}j,t} f^{T\text{-}j}{}_X[X_{T\text{-}j,t}, k_{T\text{-}j,t\text{-}1}, \sum_{i=1}^{T-j-1} Y_{T\text{-}j\text{-}i,t\text{-}i} (1 - d)^i] = 0, \; j = 0,1,2,\dots,T\text{-}1$$

(6.20) $$\frac{\partial W}{\partial Y_{T-j,t}} = - \lambda_{T\text{-}j,t} + \sum_{s=1}^{j} \lambda_{T\text{-}j+s,t+s} f^{T\text{-}j+s}{}_Y[X_{T\text{-}j+s,t+s}, k_{T\text{-}j+s,t\text{-}1+s}, \sum_{i=1}^{T-j+s-1} Y_{T\text{-}j+s\text{-}i,t\text{-}i} (1 - d)^i] (1 - d)^s = 0, \; j = 1,2,\dots,T\text{-}1$$

(6.21) $$\frac{\partial W}{\partial Y_{T,t}} = - \lambda_{T,t} + \mu_t = 0$$

(6.22) $\frac{\partial W}{\partial k_{T-j,t}} = - \mu_t (1 + n) + \mu_{t+1} (1 - d') + \lambda_{T-j,t+1} f^{T-j}_k[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1 - d)^i] = 0$, $j = 0,1,2,\ldots,T-1$

From (6.22) and (6.18)

(6.23) $\lambda_{T-j,t+1} = [(1 + \rho)^{-t} U_C(C_t) (1 + n) - (1 + \rho)^{-(t+1)} U_C(C_{t+1}) (1 - d')] / f^{T-j}_k[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1 - d)^i]$, $j = 0,1,2,\ldots T-1$

Replacing in (6.20):

(6.24) $\lambda_{T-j,t} = \sum_{s=1}^{j} [(1 + \rho)^{-(t+s-1)} U_C(C_{t+s-1}) (1 + n) - (1 + \rho)^{-(t+s)} U_C(C_{t+s}) (1 - d')] f^{T-j+s}_Y[X_{T-j+s,t+s}, k_{T-j+s,t-1+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i} (1 - d)^i] (1 - d)^s / f^{T-j+s}_k[X_{T-j+s,t+s}, k_{T-j+s,t+s-1}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s-i} (1 - d)^i]$, $j = 1,2,\ldots T-1$

Leading one period and equating to (6.23):

(6.25) $[(1 + \rho)^{-t} U_C(C_t) (1 + n) - (1 + \rho)^{-(t+1)} U_C(C_{t+1}) (1 - d')] / f^{T-j}_k[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1 - d)^i] = \sum_{s=1}^{j} [(1 + \rho)^{-(t+s)} U_C(C_{t+s}) (1 + n) - (1 + \rho)^{-(t+s+1)} U_C(C_{t+s+1}) (1 - d')] f^{T-j+s}_Y[X_{T-j+s,t+s+1}, k_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+1-i} (1 - d)^i] (1 - d)^s / f^{T-j+s}_k[X_{T-j+s,t+s+1}, k_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s+1-i} (1 - d)^i]$, $j = 1,2,\ldots T-1$

or

(6.26) $[U_C(C_t) (1 + n) - (1 + \rho)^{-1} U_C(C_{t+1}) (1 - d')] / f^{T-j}_k[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1 - d)^i] = \sum_{s=1}^{j} [(1 + \rho)^{-s} U_C(C_{t+s}) (1 + n) - (1 + \rho)^{-(s+1)} U_C(C_{t+s+1}) (1 - d')] f^{T-j+s}_Y[X_{T-j+s,t+s+1}, k_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+1-i} (1 - d)^i] (1 - d)^s / f^{T-j+s}_k[X_{T-j+s,t+s+1}, k_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t+s+1-i} (1 - d)^i]$, $j = 1,2,\ldots,T-1$

A steady-state – with $C_t = C^*$, $k_{T-j,t} = k_{T-j}^*$, $X_{T-j,t} = X_{T-j}^*$, etc. – implies that all multipliers decrease at rate ρ. (6.24) implies:

$$(6.27)\ \lambda_{T-j,t} = (1+\rho)^{-t}\, U_C(C^*)\, [(1+\rho)(1+n) - (1-d')] \,/\, f^{T-j}{}_k[X_{T-j}^*, k_{T-j}^*, \sum_{i=1}^{T-j-1} Y_{T-j-i}^* (1-d)^i]\ ,\ j = 1,2,\ldots,T-1$$

(6.25) then implies that along a balanced path:

$$(6.28)\qquad f^{T-j}{}_k[X_{T-j}^*, k_{T-j}^*, \sum_{i=1}^{T-j-1} Y_{T-j-i}^* (1-d)^i] = [(1+n) - (1+\rho)^{-1}(1-d')] \,/ \sum_{s=1}^{j} [(1+\rho)^{-s}(1+n) - (1+\rho)^{-(s+1)}(1-d')]\, f^{T-j+s}{}_Y[X_{T-j+s}^*, k_{T-j+s}^*, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i}^* (1-d)^i]\,(1-d)^s \,/\, f^{T-j+s}{}_k[X_{T-j+s}^*, k_{T-j+s}^*, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i}^* (1-d)^i]\quad ,\ j = 1,2,\ldots, T-1$$

Proposition 7: Introducing the product-chain system into the Ramsey's growth model, a steady-state may exist with constant per capita aggregates and obeys (6.28), (6.14) to (6.16).

5.2.3. Vertical Systems in Growth Models (without Capital [13]): Possible Direct Consumption of Intermediate Products.

. Finally, a multi-stage consumption felicity can be forwarded:

$$(6.29)\qquad \underset{X_{T+1}, X_{T+2},\ldots}{Max} \qquad \sum_{t=1}^{\infty} U(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})\,(1+\rho)^{-t}$$

s.t.

$$(6.30)\ Y_{T-j,t} + C_{T-j,t} = F^{T-j}[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}(1-d)^i] - \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}(1-d)^i =$$

$$= f^{T-j}[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}(1-d)^i]\ ,\ j = 1\ldots, T-1$$

$$(6.31)\qquad C_{T,t} = F^{T}[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-j-i,t-i}(1-d)^i] - \sum_{i=1}^{T-1} Y_{T-j-i,t-i}(1-d)^i =$$

$$= f^{T}[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-i,t-i}(1-d)^i]$$

[13] With Capital is reproduced in Appendix B.

(6.32) $$\sum_{j=0}^{T-1} X_{T-j,t} = X$$

(6.33) Given $Y_{1,0}, Y_{2,0},\ldots,Y_{T-1,0}$,

The problem is equivalent to:

(6.34) $$\underset{\substack{C_1,C_2,\ldots \\ X_{1,1},X_{1,2},\ldots,X_{T,1},X_{T,2},\ldots \\ Y_{1,1},Y_{1,2},\ldots,Y_{T,1},Y_{T,2},\ldots \\ \lambda_{1,1},\lambda_{1,2},\ldots,\lambda_{T,1},\lambda_{T,2},\ldots \\ \eta_1,\eta_2,\ldots}}{Max} \sum_{t=1}^{\infty} U(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) (1+\rho)^{-t} + \sum_{t=1}^{\infty} \sum_{j=1}^{T-1} \lambda_{T-j,t} \{- Y_{T-j,t} - C_{T-j,t} + f^{T-j}[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i]\} + \sum_{t=1}^{\infty} \eta_t (X - \sum_{j=0}^{T-1} X_{T-j,t}) + \sum_{t=1}^{\infty} \lambda_{T,t} \{- C_{T,t} + f^T[X_{T,t}, \sum_{i=1}^{T-1} Y_{T-j-i,t-i} (1-d)^i]\}$$

FOC yield:

(6.35) $$\frac{\partial W}{\partial C_{T-j,t}} = (1+\rho)^{-t} U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) - \lambda_{T-j,t} = 0,\ j = 0,1,2,\ldots, T-1$$

(6.36) $$\frac{\partial W}{\partial X_{T-j,t}} = -\eta_t + \lambda_{T-j,t} f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i] = 0,\ j = 0,1,2,\ldots,T-1$$

(6.37) $$\frac{\partial W}{\partial Y_{T-j,t}} = -\lambda_{T-j,t} + \sum_{s=1}^{j} \lambda_{T-j+s,t+s}\ f^{T-j+s}{}_Y[X_{T-j+s,t+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i} (1-d)^i] (1-d)^s = 0,\ j = 1,2,\ldots,T-1$$

(6.36) implies:

(6.38) $$(1+\rho)^{-t} U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i] =$$

$$= (1+\rho)^{-t} U_{C_{T-j'}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) f^{T-j'}{}_X[X_{T-j',t}, \sum_{i=1}^{T-j'-1} Y_{T-j'-i,t-i} (1-d)^i]$$

or

(6.39) $$U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) / U_{C_{T-j'}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) =$$

$$= f^{T-j'}{}_X[X_{T-j',t}, \sum_{i=1}^{T-j'-1} Y_{T-j'-i,t-i} (1-d)^i] / f^{T-j}{}_X[X_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^i] \quad , \quad j,j' = 0,1,2,\ldots,T-1$$

The MRS in consumption between two of each arguments of felicity equals the corresponding ratio of marginal products with respect to X.

Replacing (6.38) in (6.36)

(6.40) $(1 + \rho)^{-t}\ U_{C_{T-j}}(C_{T,t},\ C_{T-1,t},\ldots,\ C_{1,t}) = \sum_{s=1}^{j} (1 + \rho)^{-t-s}\ U_{C_{T-j+s}}(C_{T,t+s},\ C_{T-1,t+s},\ldots,\ C_{1,t+s})\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s},\ \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1 - d)^{i}]\ (1 - d)^{s}$ or

(6.41) $1 = \sum_{s=1}^{j} (1 + \rho)^{-s}\ [U_{C_{T-j+s}}(C_{T,t+s},\ C_{T-1,t+s},\ldots,\ C_{1,t+s}) / U_{C_{T-j}}(C_{T,t},\ C_{T-1,t},\ldots,\ C_{1,t})]\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s},\ \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1 - d)^{i}]\ (1 - d)^{s}$

Using now (6.39):

(6.42) $1 = \sum_{s=1}^{j} (1 + \rho)^{-s}\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s},\ \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1 - d)^{i}]\ (1 - d)^{s}\ f^{T-j}{}_{X}[X_{T-j,t},\ k_{T-j,t-1},\ \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1 - d)^{i}] / f^{T-j+s}{}_{X}[X_{T-j+s,t},\ \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1 - d)^{i}]$ or

(6.43) $1 = \sum_{s=1}^{j} (1 + \rho)^{-s}\,(1 - d)^{s}\ f^{T-j}{}_{X}[X_{T-j,t},\ \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1 - d)^{i}]\ \ j = 1,2,\ldots, T-1$

The real interest rate in the economy – the one at which individuals – or rather, households - discount consumption, or are willing to trade t's consumption/real goods for t+1 ones and therefore ask for to lend – is the marginal rate of substitution between ($L_{t+1}\ c_{t+1}$) and ($L_t\ c_t$) over the individuals welfare function minus 1 [14]: Now, we must choose the numeraire – the consumption good at which the exchange takes place; if the highest level consumption, $C_{T,t}$ – and production, $Y_{T,t}$, given (6.35).

$$(6.44)\quad -\left[\frac{d(L_{t+1}C_{T,t+1})}{d(L_t C_{T,t})}\right]_{|\overline{W}} - 1 = -\frac{L_{t+1}}{L_t}\left(\frac{dC_{T,t+1}}{dC_{T,t}}\right)_{|\overline{W}} - 1 = (1 + n)\left(\frac{\dfrac{\partial W}{\partial C_{T,t}}}{\dfrac{\partial W}{\partial C_{T,t+1}}}\right) - 1 =$$

[14] In the Ramsey's model, the real rate of return to savings is – equated to - $r_t = f'(k_{t-1}) - d' = R_t/P_t - d'$ – see Barro and Sala-i-Martin (1995), p. 63-69. The term $(1 - d')\ (P_{k,t} - P_{k,t-1}) / P_{k,t-1}$ should be added – see footnote 11. p. 69 of the same reference - when $P_{k,t}$ is the price of capital in units of consumables – in the one-sector model, fixed to 1.

$$= (1+n) \frac{(1+\rho)^{-t} U_{C_T}(C_{T,t}, C_{T-1,t}, \ldots, C_{1,t})}{(1+\rho)^{-(t+1)} U_{C_T}(C_{T,t+1}, C_{T-1,t+1}, \ldots, C_{1,t+1})} - 1 = (1+n)(1+\rho) \frac{U_{C_T}(C_{T,t}, C_{T-1,t}, \ldots, C_{1,t})}{U_{C_T}(C_{T,t+1}, C_{T-1,t+1}, \ldots, C_{1,t+1})} - 1 = (1+\rho') \frac{U_{C_T}(C_{T,t}, C_{T-1,t}, \ldots, C_{1,t})}{U_{C_T}(C_{T,t+1}, C_{T-1,t+1}, \ldots, C_{1,t+1})} - 1 = r_{T,t}$$

where (1 + ρ) was replaced by (1 + ρ) = (1 + ρ') / (1 + n) and ρ' denotes the individuals' discount rate when future household members are valued and population grows at rate n per period. Again, if we consider $r_{j,t}$ j = 1,2,…,T, the term structure of those interest rates can be increasing in j – and we observe a decreasing pattern of interest rates relative to maturity - if $\frac{U_{C_j}(C_{T,t}, C_{T-1,t}, \ldots, C_{1,t})}{U_{C_j}(C_{T,t+1}, C_{T-1,t+1}, \ldots, C_{1,t+1})}$ increases with j. In a steady-state path with $C_{j,t} = C_j^*$, the interest rate(s) will be constant and equal ρ'.

Proposition 8: Introducing the product-chain system into the one-sector growth model (without capital), and allowing intermediate products to also be consumed:

8.1 a steady-state may exist and obeys (6.41), (6.43) and (6.30) to (6.32).

8.2 interest rates are indexed to a particular (tier) intermediate product.

Conclusion

Intertemporal preferences embedding memory effects were allowed to replace standard static felicity functions. Discounting – constant discounting – and additive accumulation – first-order separability - is still assumed; consistency is therefore not threatened by the assumption.

The intertemporal optimization problem yields interesting interpretations. Decisions become less fluid.

When preferences are coupled with capital accumulation, the equilibrium interest rate in the economy may be the same as in the standard case – but economies may prefer, in case of consumption-past welfare complementarity, a larger stock of capital. Balanced paths are no longer necessarily compatible with steady-state constant levels of per capita consumption – much less felicity that here exhibits a durable good nature.

Possible extension with variable discount and depreciation/forgetfulness rates are not expect to alter the problem solution –as long as each periodic rate is time specific and accumulating in product terms – guaranteeing time-consistency.

Models of product-chain systems use the same type of mechanics. They were shown to have potential consistency problems and suggest an explanation for a term structure of interest rates decreasing with distance to maturity (here, final product stage). Embedded in general equilibrium setups – as Ramsey's growth model – generate possible steady-states.

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Appendix A

Let an individual be born at time 0; U_t denotes his felicity in period t, and he is endowed with a stream U_t, t = 1,2,…,n. If he discounts the future at (heterogeneous) periodic rates r_t, at time 0, his prospective welfare is:

$$\text{(A.1)} \quad W_0 = \sum_{t=1}^{\infty} \frac{U_t}{\prod_{i=1}^{t}(1+r_i)} = \sum_{t=1}^{T} \frac{U_t}{\prod_{i=1}^{t}(1+r_i)} + \frac{1}{\prod_{i=1}^{T}(1+r_i)} \sum_{t=1}^{\infty} \frac{U_{T+t}}{\prod_{i=1}^{t}(1+r_{T+i})}$$

Standing at time T > 0, the individual's welfare prospects are $W_T = \sum_{t=1}^{\infty} \frac{U_{T+t}}{\prod_{i=1}^{t}(1+r_{T+i})}$;

his evaluation, at that moment T, of all his life's initial potential is:

$$\text{(A.2)} \quad W_0^T = W_0 \prod_{i=1}^{T}(1+r_i) = \sum_{t=1}^{T} U_t \prod_{i=t+1}^{T}(1+r_i) + \sum_{t=1}^{\infty} \frac{U_{T+t}}{\prod_{i=1}^{t}(1+r_{T+i})} = \sum_{t=1}^{T} U_t \prod_{i=t+1}^{T}(1+r_i) + W_T =$$

$$\text{(letting j = T – t:)} = \sum_{j=0}^{T-1} U_{T-j} \frac{\prod_{i=0}^{j}(1+r_{T-i})}{1+r_{T-j}} + W_T = \sum_{j=1}^{T} U_{T-j+1} \frac{\prod_{i=0}^{j-1}(1+r_{T-i+1})}{1+r_{T-j}} + W_T$$

Exemplifying,

$$W_0^4 = U_1 (1 + r_2) (1 + r_3) (1 + r_4) + U_2 (1 + r_3) (1 + r_4) + U_3 (1 + r_4) + U_4 + W_4$$

where $W_4 = \sum_{t=1}^{\infty} \frac{U_{4+t}}{\prod_{i=1}^{t}(1+r_{4+i})}$, accumulated prospects at moment 4, consistent with W_0. Then we have a compatible summation writing.

Then, $W_T = W_0^T - \sum_{j=0}^{T-1} U_{T-j} \frac{\prod_{i=0}^{j}(1+r_{T-i})}{1+r_{T-j}}$. All else (including total life's prospects) fixed, $-\frac{dW_T}{dU_{T-j}}$ is how much of an increase of current – time T's - prospects he would have to be (have been…) given to let go of one unit of a good memory of time (T – j). Using the last expression,

$$(A.3)\quad -\frac{dW_T}{dU_{T-j}} = \frac{\prod_{i=0}^{j}(1+r_{T-i})}{1+r_{T-j}} = (1 + r_T)\,(1 + r_{T-1}) \ldots (1 + r_{T-j+1}) = \prod_{i=T-j+1}^{T}(1+r_i)$$

Such "price" of units of U_{T-j} for an individual located at T – the value at T of U_{T-j} - is larger than 1 and increases with j – with how distant in the past (T – j) is of T… So discounting mimics an increasing value and, therefore, effect (positive or negative: the argument would also apply to losses…) on current decisions, of previous remembrances as these become more distant in the past.

Also, as

$$(A.4)\qquad W_0 = \frac{1}{\prod_{i=1}^{T}(1+r_i)} \sum_{j=0}^{T-1} U_{T-j}\frac{\prod_{i=0}^{j}(1+r_{T-i})}{1+r_{T-j}} + \frac{1}{\prod_{i=1}^{T}(1+r_i)} W_T = \frac{1}{\prod_{i=1}^{T'}(1+r_i)}$$

$$\sum_{j=0}^{T'-1} U_{T'-j}\frac{\prod_{i=0}^{j}(1+r_{T'-i})}{1+r_{T'-j}} + \frac{1}{\prod_{i=1}^{T'}(1+r_i)} W_{T'}$$

the last equality suggests how accumulated prospects are relatively valued along lifetime indifference curves; if total wealth changes and that is not to affect U_t, t ≤ Max(T, T'), the relative impact on accumulated prospects at the two points in time is

$$(A.5)\qquad \frac{\frac{dW_0}{dW_{T'}}}{\frac{dW_0}{dW_T}} = \frac{dW_T}{dW_{T'}} = \frac{\prod_{i=1}^{T}(1+r_i)}{\prod_{i=1}^{T'}(1+r_i)} = \prod_{i=T'+1}^{T}(1+r_i)$$

and we can write

$$(A.6)\qquad \frac{dW_T}{dW_{T-j}} = \prod_{i=T-j+1}^{T}(1+r_i) = -\frac{dW_T}{dU_{T-j}} \quad ,\ \text{for } j > 0$$

Appendix B

5.2.3. Vertical Systems in Growth Models (with Capital): Possible Direct Consumption of Intermediate Products.

. Finally, a multi-stage consumption felicity can be forwarded:

$$(6.29) \qquad \underset{X_{T+1},X_{T+2},\ldots,k_{T+1},k_{T+2},\ldots}{Max} \sum_{t=1}^{\infty} U(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})\,(1+\rho)^{-t}$$

s.t.

$$(6.30) \quad Y_{T-j,t} + C_{T-j,t} = F^{T-j}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] + \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i} =$$

$$= f^{T-j}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}]\,, \quad j = 0,1\ldots, T-1$$

$$(6.31) \qquad \sum_{j=0}^{T-1} X_{T-j,t} = X$$

$$(6.32) \qquad (1+n)\sum_{j=0}^{T-1} k_{T-j,t} = (1-d')\sum_{j=0}^{T-1} k_{T-j,t-1} + Y_{T,t} - C_{T,t}$$

Given $k_{1,0}, k_{2,0},\ldots,k_{T,0}$ and $Y_{1,0}, Y_{2,0},\ldots,Y_{T-1,0}$

The problem is equivalent to:

$$(6.33) \quad \underset{\substack{C_1,C_2,\ldots \\ X_{1,1},X_{1,2},\ldots,X_{T,1},X_{T,2},\ldots \\ k_{1,1},k_{1,2},\ldots,k_{T,1},k_{T,2},\ldots \\ Y_{1,1},Y_{1,2},\ldots,Y_{T,1},Y_{T,2},\ldots \\ \lambda_{1,1},\lambda_{1,2},\ldots,\lambda_{T,1},\lambda_{T,2},\ldots \\ \mu_1,\mu_2,\ldots \\ \eta_1,\eta_2,\ldots}}{Max} \sum_{t=1}^{\infty} U(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})\,(1+\rho)^{-t} + \sum_{t=1}^{\infty}\sum_{j=0}^{T-1} \lambda_{T-j,t}\,\{- Y_{T-j,t} -$$

$$C_{T-j,t} + f^{T-j}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}]\} + \sum_{t=1}^{\infty} \eta_t\,(X - \sum_{j=0}^{T-1} X_{T-j,t}) + \sum_{t=1}^{\infty} \mu_t\,[(1-d')$$

$$\sum_{j=0}^{T-1} k_{T-j,t-1} + Y_{T,t} - C_{T,t} - (1+n)\sum_{j=0}^{T-1} k_{T-j,t}]$$

FOC yield:

$$(6.34) \quad \frac{\partial W}{\partial C_{T,t}} = (1+\rho)^{-t}\, U_{C_T}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) - \mu_t = 0$$

$$(6.35) \quad \frac{\partial W}{\partial C_{T-j,t}} = (1+\rho)^{-t}\, U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) - \lambda_{T-j,t} = 0, \; j = 1,2,\ldots, T-1$$

$$(6.36) \quad \frac{\partial W}{\partial X_{T-j,t}} = -\eta_t + \lambda_{T-j,t}\, f^{T-j}_{X}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] = 0, \; j =$$

0,1,…,T-1

$$(6.37) \quad \frac{\partial W}{\partial Y_{T,t}} = -\lambda_{T,t} + \mu_t = 0$$

(6.38) $\dfrac{\partial W}{\partial Y_{T-j,t}} = -\lambda_{T-j,t} + \sum_{s=1}^{j} \lambda_{T-j+s,t+s}\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s}, k_{T-j+s,t-1+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]\ (1-d)^{s} = 0$, $j = 1,2,\ldots, T-1$

(6.39) $\dfrac{\partial W}{\partial k_{T-j,t}} = -\mu_{t}\,(1+n) + \mu_{t+1}\,(1-d') + \lambda_{T-j,t+1}\ f^{T-j}{}_{k}[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i}\,(1-d)^{i}] = 0$, $j = 0,1,2,\ldots,T-1$

(6.35), (6.36) and (6.37) imply:

(6.40) $(1+\rho)^{-t}\, U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})\ f^{T-j}{}_{X}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] =$

$= (1+\rho)^{-t}\, U_{C_{T-j'}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})\ f^{T-j'}{}_{X}[X_{T-j',t}, k_{T-j',t-1}, \sum_{i=1}^{T-j'-1} Y_{T-j'-i,t-i}\,(1-d)^{i}]$

or

(6.41) $U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) / U_{C_{T-j'}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) =$

$= f^{T-j'}{}_{X}[X_{T-j',t}, k_{T-j',t-1}, \sum_{i=1}^{T-j'-1} Y_{T-j'-i,t-i}\,(1-d)^{i}] / f^{T-j}{}_{X}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}]$, $j,j' = 0,1,2,\ldots,T-1$

The MRS in consumption between two of each arguments of felicity equals the corresponding ratio of marginal products with respect to X.

Replacing (6.34) to (6.37) in (6.38):

(6.42) $(1+\rho)^{-t}\, U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) = \sum_{s=1}^{j} (1+\rho)^{-t-s}\, U_{C_{T-j+s}}(C_{T,t+s}, C_{T-1,t+s},\ldots, C_{1,t+s})\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s}, k_{T-j+s,t-1+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]\ (1-d)^{s}$ or

(6.43) $1 = \sum_{s=1}^{j} (1+\rho)^{-s}\, [U_{C_{T-j+s}}(C_{T,t+s}, C_{T-1,t+s},\ldots, C_{1,t+s}) / U_{C_{T-j}}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t})]\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s}, k_{T-j+s,t-1+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]\ (1-d)^{s}$

Using now (6.41):

(6.44) $1 = \sum_{s=1}^{j} (1+\rho)^{-s}\ f^{T-j+s}{}_{Y}[X_{T-j+s,t+s}, k_{T-j+s,t-1+s}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]\ (1-d)^{s}\ f^{T-j}{}_{X}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i}\,(1-d)^{i}] / f^{T-j+s}{}_{X}[X_{T-j+s,t}, k_{T-j+s,t-1}, \sum_{i=1}^{T-j+s-1} Y_{T-j+s-i,t-i}\,(1-d)^{i}]$ or

$$(6.45)\quad 1 = \sum_{s=1}^{j} (1+\rho)^{-s} (1-d)^{s} \; f^{T-j}{}_{X}[X_{T-j,t}, k_{T-j,t-1}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t-i} (1-d)^{i}] \quad , j = 1,2,\ldots, T-1$$

Using (6.34) to (6.37) and (6.39):

$$(6.46)\quad (1+\rho)^{-t} U_{C_T}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) \; (1+n) + (1+\rho)^{-t} U_{C_T}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) \; (1-d') + (1+\rho)^{-t-1} U_{C_{T-j-1}}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) \; f^{T-j}{}_{k}[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1-d)^{i}] = 0 \; , j = 1,2,\ldots,T-1$$

or

$$(6.47)\quad [U_{C_T}(C_{T,t}, C_{T-1,t},\ldots, C_{1,t}) / U_{C_T}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1})] (1+\rho)(1+n) = -(1+\rho)(1-d') + [U_{C_{T-j-1}}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) / U_{C_T}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1})] \; f^{T-j}{}_{k}[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1-d)^{i}] = 0 \; , j = 1,2,\ldots,T-1$$

Then: $U_{C_{T-j-1}}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) \; f^{T-j}{}_{k}[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1-d)^{i}]$ will be constant for j = 1,2,…,T-1:

$$(6.48)\quad U_{C_{T-j-1}}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) / U_{C_{T-j'-1}}(C_{T,t+1}, C_{T-1,t+1},\ldots, C_{1,t+1}) = f^{T-j'}{}_{k}[X_{T-j',t+1}, k_{T-j',t}, \sum_{i=1}^{T-j'-1} Y_{T-j'-i,t+1-i} (1-d)^{i}] \; / \; f^{T-j}{}_{k}[X_{T-j,t+1}, k_{T-j,t}, \sum_{i=1}^{T-j-1} Y_{T-j-i,t+1-i} (1-d)^{i}] \quad , \; j,j' = 1,2,\ldots,T-1$$

The marginal rate of substitution equals the marginal rate of transformation between any two intermediate products.

The real interest rate in the economy – the one at which individuals – or rather, households - discount consumption, or are willing to trade t's consumption/real goods for t+1 ones and therefore ask for to lend – is the marginal rate of substitution between ($L_{t+1}\, c_{t+1}$) and ($L_t\, c_t$) over the individuals welfare function minus 1 [15]: Now, we must choose the numeraire – the

[15] In the Ramsey's model, the real rate of return to savings is – equated to - $r_t = f'(k_{t-1}) - d = R_t/P_t - d$ – see Barro and Sala-i-Martin (1995), p. 63-69. The term $(1-d)\,(P_{k,t} - P_{k,t-1}) / P_{k,t-1}$ should be added – see footnote 11. p. 69 of the same reference - when $P_{k,t}$ is the price of capital in units of consumables – in the one-sector model, fixed to 1.

consumption good at which the exchange takes place; if the highest level consumption, $C_{T,t}$ – and production, $Y_{T,t}$, given (6.31).

$$(6.49)\quad -\left[\frac{d(L_{t+1}C_{T,t+1})}{d(L_tC_{T,t})}\right]_{|\overline{W}} - 1 = -\frac{L_{t+1}}{L_t}\left(\frac{dC_{T,t+1}}{dC_{T,t}}\right)_{|\overline{W}} -1 = (1+n)\left(\frac{\frac{\partial W}{\partial C_{T,t}}}{\frac{\partial W}{\partial C_{T,t+1}}}\right) - 1 =$$

$$= (1+n)\frac{(1+\rho)^{-t}U_{C_T}(C_{T,t},C_{T-1,t},...,C_{1,t})}{(1+\rho)^{-(t+1)}U_{C_T}(C_{T,t+1},C_{T-1,t+1},...,C_{1,t+1})} - 1 = (1+n)(1+\rho)\frac{U_{C_T}(C_{T,t},C_{T-1,t},...,C_{1,t})}{U_{C_T}(C_{T,t+1},C_{T-1,t+1},...,C_{1,t+1})} - 1 = (1+\rho')\frac{U_{C_T}(C_{T,t},C_{T-1,t},...,C_{1,t})}{U_{C_T}(C_{T,t+1},C_{T-1,t+1},...,C_{1,t+1})} - 1 = r_{T,t}$$

where (1 + ρ) was replaced by (1 + ρ) = (1 + ρ') / (1 + n) and ρ' denotes the individuals' discount rate when future household members are valued and population grows at rate n per period. It has correspondence with (6.44). In a steady-state path for $C_{j,t} = C_j^*$, the interest rate will be constant.

Proposition 8: Introducing the product-chain system into the Ramsey's growth model, and allowing intermediate products to also be consumed:

8.1 a steady-state may exist and obeys (6.41), (6.47), (6.30) to (6.32).

8.2 interest rates are indexed to a particular (tier) intermediate product.