EERI
Economics and Econometrics Research Institute

From Fault Tree to Credit Risk Assessment: A Case Study

Hayette Gatfaoui

EERI Research Paper Series No 5/2004

EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 299 3523
Fax: +322 299 3523
www.eeri.eu

From Fault Tree to Credit Risk Assessment: A Case Study

Dr. Hayette Gatfaoui

Associate Professor, Rouen School of Management, Economics & Finance Department, 1 Rue du Maréchal Juin, BP 215, 76825 Mont-Saint-Aignan Cedex France; Phone: 0033232825700, Fax: 0033232825833; hayette.gatfaoui@groupe-esc-rouen.fr*
and Quantitative Finance Research Centre, School of Finance and Economics, University of Technology, Sydney.

First Draft: February 2004; This Draft: September 2004

Abstract

Reliability has been largely applied to industrial systems in order to study the various possibilities of sytems' failure. The goal is to establish the chain of events leading to any system's failure, namely the top event. Looking for the minimal paths leading to any system's fault allows for a better control of systems' safety. To this end, reliability is composed of a static approach (see Ngom et al. [1999] for example) as well as a dynamic approach (see Reory & Andrews [2003] for example). In this paper, we extend the framework stated by Gatfaoui (2003) allowing for the application of fault tree theory to credit risk assessment. The author explains that fault tree is one alternative approach of reliability, which matches default risk analysis in a simple framework. Our extension includes other distributions of probability to model the lifetimes of French firms while studying the related empirical default probabilities. We use mainly, but not exclusively, continuous distributions for which the exponential law used by Gatfaoui (2003) constitutes a particular case. Our results exhibit both the exponential nature of French firms' lifetimes as well as strong convex and fast decreasing time varying failure rates. Such a feature has some non-negligible impact insofar as it characterizes corresponding credit spreads' term structure.

Keywords : credit risk, default probability, failure rate, fault tree, reliability, survival.
JEL codes : C1, D8.

*This email address is not working now, please send your message to h.gatfaoui@easynet.fr or gatfaoui@univ-paris1.fr .

1 Introduction

Whatever the considered matter, risk has become of major interest since the 80' (see Cabarbaye [1998], Henley & Kumamoto [1992] and Papoulis [1984] for example). The risk that a disastrous event occurs is of great importance since such an event engenders social harm as well as economic and financial losses. This principle also applies to credit risk valuation, which has been focused since the last decade. Indeed, the Basel II directives underline the importance of the ability to value and quantify fairly default risk (see Basel Committee on banking supervision [1996] for example). Therefore, the sound and reliable assessment of default risk represents the challenge of the next decade. Along with this consideration, we employ the simple setting of Gatfaoui (2003) to value credit risk. The author applies fault tree theory to assess default risk in a simple framework (see Bon [1995] and Rothenthal [1998] among others). More precisely, starting from the empirical default probabilities characterizing the bankruptcy of French firms, fault tree analysis allows for estimating the hazard rates, or equivalently, the failure rates of these French firms. The process is easy and straightforward. The lifetimes of French firms are the focus of the study since any firm's default probability corresponds to the probability of death of the firm, or equivalently, to the probability that its lifetime ends. Therefore, failure rates' estimations depend on the probability distribution of the related lifetimes. Gatfaoui (2003) chose to resort to an exponential law with a constant intensity in order to describe French firms' default probabilities. However, although this statistical representation seems to be appropriate, the author finds that the corresponding implied failure rates are time varying, and exhibit a convex decreasing pattern. Such a result is in accordance with the work of Fons & Kimball (1991) who highlight the significant time varying behavior of failure rates. This time varying feature is shown to be as important as firms' credit ratings are low. Moreover, choosing an exponential law with a constant parameter implicitly assumes a time independence for the hazard rate function (i.e., the present does not depend on the past). Such an assumption is nevertheless inconsistent with modern default risk analysis. Indeed, it is well known that bankruptcy threatens especially young firms under five years old, supporting the existence of a life cycle for firms. Such a consideration suggests a time dependence for the hazard function. Namely, hazard rates or, equivalently, default probabilities should have higher levels at the beginning of newly created firms' existence.

Although a time varying intensity exponential law can be proxied by a series of constant intensity exponential laws over well chosen and sufficiently small time subsets, we focus on the global behavior of hazard rates. We propose consequently to extend the work of Gatfaoui (2003) in order to take into account the time dependence of the failure rate function. For this purpose, we consider the following set of probability distributions, namely lognormal, log-logistic, gamma, weibull, beta of second species, a mixture of two exponential laws with constant intensity, and finally, two non-homogeneous Poisson processes known as Cox-Lewis and exponential exponent. In this way, we are able to account for a wide range of whether monotonous, hump-shaped, convex or concave failure rates relative to time. And, we can capture most of empirical well known patterns describing corporate failures. We hope our framework to allow for parameter estimates leading to distributions for which the two first moments are defined and finite. Indeed, the existence of the first two moments is extremely important in characterizing reliability (i.e., survival time of firms). Specifically, our distributions' set requires the existence and boundedness of their respective mean and variance (i.e., volatility). These two moments belong to the key parameters and conditions that define each of our eight possible probability distributions in a theoretical viewpoint (e.g., mean time to failure).

Our paper is organized as follows. First, we recall the theoretical framework for reliability (i.e., basic notions and principles) and the characteristics of each probability distribution (i.e., statistical properties). Second, we present the related results (e.g., parameter estimates for each distribution-type) and the Kolmogorov adequacy test ensuring the soundness of our representations. We also perform an exponentiality test to investigate the coherency of our non-homogeneous Poisson processes versus the classical exponential law with constant intensity while describing French failures. Such a test allows us to investigate the usefulness and relevance of a time varying intensity parameter versus a constant one in exponential-type representations. Third, we look for the optimal representation of our failure rates given our set of consistent probability distributions, and compute the related forward conditional default probabilities over various time horizons. The optimality criterion we employ solves a quadratic problem, namely the minimization of some absolute error function. Hence, the optimal characterization fits at best the empirical default probabilities under consideration. Fourth, we use the obtained optimal representations to deduce the corresponding term structure of credit risky discount bonds. By the way, we underline the link prevailing between the reduced form approach of credit risk and reliability. Precisely, the reduced form approach is known to often stipulate *a priori* dynamics for risky bonds' term structure and therefore credit spreads' term structure. This branch of credit risk assessment is based on the study of the default time's arrival, and its representation as a random variable since the instant of potential default is unknown and uncertain. Such a setting is therefore founded on the intensity process of default, which describes the probability that a default event occurs over any infinitesimal time interval. Hence, characterizing credit spreads' term structure requires only information about both the hazard rate function and the corresponding potential recovery rate (when the risk free term structure is deterministic at most). Finally, we end our paper with some concluding remarks and possible extensions to our analysis in the lens of time dependence and business cycle's impact. Specifically, economic world's changes impact default risk (i.e, possibility of corporate failures at any time) as time elapses. Recall that any risk profile is defined by two main dimensions, namely time and uncertainty.

2 Theoretical framework

In this section, we introduce briefly the setting employed by Gatfaoui (2003) to match fault tree analysis with credit risk assessment. Then, we present our set of chosen probability distributions aimed at describing any default event.

2.1 Fault tree theory

Fault tree theory requires to model the lifetime X of the system under consideration, namely any firm here. The basic assumptions concern an elementary fault tree[1] and state that the firm's debt outstandings (e.g., some solvency principles) engender the default event, which occurs suddenly without any possibility to recover from the failure state. Moreover, any firm is assumed to be always in a non-default state before default occurs. In such a case, the default probability p_t at current time t corresponds to the probability that the firm's lifetime ends between 0 and t. Therefore, the default probability depends strongly on the cumulative distribution function[2] F, and consequently, the distribution function[3] f of the lifetime of any firm since $p_t = F(t) = P(X \leq t) = \int_0^t f(s)\, ds$ for each time $t > 0$. In a symmetric way, the survival function R corresponds to the probability that the firm's lifetime still goes on after a given date, namely $R(t) = 1 - F(t) = P(X > t) = \int_t^{+\infty} f(s)\, ds$ for each time $t > 0$. And, the lifetime's first moment for example then writes:

$$E[X] = \int_0^{+\infty} s\, f(s)\, ds \tag{1}$$

Such a setting allows us to define the hazard rate function λ, or equivalently, the failure rate as follows for each time $t > 0$:

$$\lambda(t) = \lim_{\Delta t \to 0^+} \frac{P(t < X \leq t + \Delta t \mid X > t)}{\Delta t} \tag{2}$$

which implies for each time $t > 0$ that:

$$R(t) = \exp\left[-\int_0^t \lambda(s)\, ds\right] \tag{3}$$

and

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)} \tag{4}$$

The hazard rate is closely linked to the probability that the lifetime of a firm ends on a specified time subset given that the firm has not defaulted before the

[1] We assume that one default event triggers the firm's bankruptcy (i.e., a simple one branch tree).

[2] It is assumed that $F(0) = 0$ and $F(+\infty) = 1$. We further assume that F is continuous and once derivable relative to time.

[3] Given our framework, f is only defined for positive values since a lifetime can only be positive.

lower bound of this time interval. Moreover, considering relation (3), we find that:

$$p_t = 1 - R(t) = 1 - \exp\left[-\int_0^t \lambda(s)\, ds\right] \tag{5}$$

Such a representation assumes explicitly a link between the default probability and its related hazard rate. We can also translate this consideration into the fact that the hazard rate impacts the default probability. And, such a feature is in accordance with the work of Bakshi et al. (2004). Therefore, starting from default probabilities, or equivalently, failure rates, we are able to characterize any firm's bankruptcy given our framework and assumptions. To this end, we just have to select the type of the probability distribution characterizing the firm's lifetime. Our related choice is introduced in the next subsection.

2.2 Statistical distributions

We present here our set of distributions aimed at describing the lifetime of any firm. Our distribution set is composed of eight probability-type functions,[4] which we introduce therein for each time $t > 0$. Our choice is driven by stylized facts that usually describe corporate default risk such as asymmetric patterns (i.e., high or low risk of loss for investors as a function of firms' creditworthiness through time).

The lognormal distribution with parameters $\mu \in \mathbb{R}$ and σ (with $\sigma > 0$) exhibits the following features for $N(.)$ being the cumulative distribution function of the standard normal law:

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\ln(t)-\mu}{\sigma}\right)^2} \tag{6}$$

$$F(t) = N\left(\frac{\ln(t) - \mu}{\sigma}\right) \tag{7}$$

Given relation (4), we know that the hazard rate function is a concave function of time such that $\lambda(0) = 0$ and $\lim_{t\to\infty} \lambda(t) = 0$. Indeed, the failure rate reads as follows for $n(.)$ being the distribution function of the standard normal law:

$$\lambda(t) = \frac{1}{t\sigma} \frac{n\left(\frac{\ln(t)-\mu}{\sigma}\right)}{1 - N\left(\frac{\ln(t)-\mu}{\sigma}\right)} \tag{8}$$

The log-logistic distribution with parameters $\mu \in \mathbb{R}$ and σ (with $\sigma > 0$) is featured as follows:

$$f(t) = \frac{1}{t\sigma} f_L\left(\frac{\ln(t) - \mu}{\sigma}\right) \tag{9}$$

$$F(t) = F_L\left(\frac{\ln(t) - \mu}{\sigma}\right) \tag{10}$$

[4] The reader is invited to consult the book of Tassi (1992) for more details.

with

$$f_L(t) = F_L(t)(1 - F_L(t)) \tag{11}$$

$$F_L(t) = \frac{1}{1 + e^{-t}} \tag{12}$$

Relation (4) allows then to express the hazard rate function such that:

$$\lambda(t) = \frac{c}{\sigma}\frac{t^{\frac{1}{\sigma}-1}}{1 + c\,t^{\frac{1}{\sigma}}} \tag{13}$$

where

$$c = e^{-\frac{\mu}{\sigma}} \tag{14}$$

Such a representation is very flexible since the failure rate can take different shapes. For example, it is hump-shaped, slightly convex and strongly convex respectively when we successively have[5] $\sigma < 1$, $\sigma = 1$ and $\sigma > 1$.

The gamma distribution with parameters α and β, namely $\Gamma(\alpha, \beta)$, is described by its scale parameter $\alpha > 0$ and its shape parameter $\beta > 0$ such that:

$$f(t) = \frac{\alpha}{\Gamma(\beta)}(\alpha t)^{\beta-1} e^{-\alpha t} \tag{15}$$

where

$$\Gamma(\beta) = \int_0^{+\infty} x^{\beta-1} e^{-x} dx = (\beta - 1)\Gamma(\beta - 1) \tag{16}$$

for any real positive β. When $\beta = 1$, the gamma distribution simply corresponds to an exponential law with a constant intensity equal to α. When $\beta \neq 1$, there is no analytical expression[6] for the hazard function, which is expressed as:

$$\lambda(t) = \frac{1}{\displaystyle\int_0^{+\infty} \left(\frac{s}{t} + 1\right)^{\beta-1} e^{-\alpha s} ds} \tag{17}$$

In this case, for $\beta > 1$, we know that the failure rate λ is a concave increasing function of time such that $\lim_{t \to 0} \lambda(0) = 0$ and $\lim_{t \to \infty} \lambda(t) = \alpha$. For $\beta < 1$, the failure rate becomes a convex decreasing function of time with $\lim_{t \to 0} \lambda(0) = \infty$ and $\lim_{t \to \infty} \lambda(t) = \alpha$. This representation[7] allows then for monotonous failure rate's behavior relative to time. Assessing the impact of the hazard rate on both recovery rates and default probabilities while studying default risk assessment, Bakshi et al. (2004) use a gamma distribution to characterize the hazard rate.

[5] Here, σ is a shape parameter.

[6] There is also no analytical expression for the cumulative distribution function F, or equivalently, the default probability and the survival function R.

[7] Recall that when $\alpha = 1$, this distribution is called 'standard gamma distribution'.

The weibull distribution with $\eta > 0$, $\beta > 0$ and $\gamma \geq 0$ parameters is characterized by its scale parameter η, its shape parameter β and its location parameter γ. We have the following description:

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}} \tag{18}$$

$$F(t) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}} \tag{19}$$

such that the failure rate then writes:

$$\lambda(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} \tag{20}$$

Notice that when lifetime X follows a weibull distribution with parameters (η, β, γ), then $\left(\frac{X-\gamma}{\eta}\right)^{\beta}$ follows an exponential law with an intensity parameter equal to 1.

The second species beta distribution $\beta(p, q)$ with parameters $p > 0$ and $q > 0$ exhibits the following distribution function:

$$f(t) = \frac{1}{B(p,q)} \frac{t^{p-1}}{(1+t)^{p+q}} \tag{21}$$

where

$$B(p,q) = B(q,p) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{22}$$

such that we get

$$F(t) = \frac{1}{B(p,q)} \int_0^t \frac{s^{p-1}}{(1+s)^{p+q}}\,ds \tag{23}$$

Here, p and q are two shape parameters, which have to be compared to 1. Indeed, their location relative to the unit value determines the degree of curvature of the failure rate.[8]

Using a mixture of two exponential laws with constant intensity assumes that the studied firms are composed of two sub-populations, namely two groups of firms whose lifetimes follow two distinct exponential laws with constant intensities. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be respectively the two intensities of our exponential processes, and π_1 and π_2 the corresponding probabilities that each sub-population follows one given exponential distribution. In this case, we have $\pi_1 + \pi_2 = 1$ such that the distribution and cumulative distribution functions of our firms' lifetimes then write:

$$f(t) = \pi_1\lambda_1 e^{-\lambda_1 t} + \pi_2\lambda_2 e^{-\lambda_2 t} = \pi_1\lambda_1 e^{-\lambda_1 t} + (1-\pi_1)\lambda_2 e^{-\lambda_2 t} \tag{24}$$

[8] It also determines the degree of curvature of the related distribution function f.

$$F(t) = 1 - \pi_1 e^{-\lambda_1 t} - \pi_2 e^{-\lambda_2 t} = 1 - \pi_1 e^{-\lambda_1 t} - (1 - \pi_1) e^{-\lambda_2 t} \tag{25}$$

Therefore, the failure rate takes the following form:

$$\lambda(t) = \frac{\pi_1 \lambda_1 + \pi_2 \lambda_2 e^{-(\lambda_2 - \lambda_1)t}}{\pi_1 + \pi_2 e^{-(\lambda_2 - \lambda_1)t}} = \frac{\pi_1 \lambda_1 + (1 - \pi_1) \lambda_2 e^{-(\lambda_2 - \lambda_1)t}}{\pi_1 + (1 - \pi_1) e^{-(\lambda_2 - \lambda_1)t}} \tag{26}$$

When $\lambda_2 > \lambda_1$, the failure rate becomes a homographic function of $e^{-(\lambda_2 - \lambda_1)t}$ and also a convex decreasing function of time with values[9] $\lambda(0) = \frac{\pi_1 \lambda_1 + \pi_2 \lambda_2}{\pi_1 + \pi_2} = \pi_1 (\lambda_1 - \lambda_2) + \lambda_2$ and $\lim_{t \to \infty} \lambda(t) = \lambda_1$. This representation is consistent with the behavior of French firms' implied failure rates exhibited by Gatfaoui (2003). Besides, such a time-varying behavior of the failure rate leads us to focus on two particular non-homogeneous Poisson processes, namely Poisson laws with time-varying intensity parameters. The first one is the Cox-Lewis process also called log-linear model with parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, and its features are as follows:

$$f(t) = \begin{cases} \exp(-e^{\alpha} t) \times e^{\alpha} \text{ if } \beta = 0 \\ \exp\left\{-\frac{e^{\alpha}}{\beta}\left(e^{\beta t} - 1\right)\right\} \times e^{\alpha + \beta t} \text{ if } \beta \neq 0 \end{cases} \tag{27}$$

$$F(t) = \begin{cases} 1 - \exp(-e^{\alpha} t) \text{ if } \beta = 0 \\ 1 - \exp\left\{-\frac{e^{\alpha}}{\beta}\left(e^{\beta t} - 1\right)\right\} \text{ if } \beta \neq 0 \end{cases} \tag{28}$$

such that we get

$$\lambda(t) = e^{\alpha + \beta t} \tag{29}$$

Notice that the case $\beta = 0$ represents the classical exponential law with parameter e^{α}. Moreover, when $\beta > 0$, the system, or equivalently, any firm is said to deteriorate whereas the firm is said to improve, or equivalently, to become economically and financially healthier when $\beta < 0$.[10]

The second non-homogeneous Poisson process corresponds to the exponential exponent law with parameters $a > 0$ and $b > 0$, which is described by:

$$f(t) = \exp\left(-at^b\right) \times ab\, t^{b-1} \tag{30}$$

$$F(t) = 1 - \exp\left(-at^b\right) \tag{31}$$

$$\lambda(t) = ab\, t^{b-1} \tag{32}$$

When $b = 1$, we get the classical exponential distribution with parameter a. Moreover, when $b < 1$, the failure rate is a convex decreasing function of time whereas it becomes an increasing (convex or concave) function of time when $b > 1$. This representation is also called 'weibull law with two parameters' and allows for monotonous failure rates relative to time.

[9] Notice that under our assumption, we have $0 < \lambda_1 \leq \lambda(0) \leq \lambda_2$.

[10] The hazard rate function is a convex function of time whatever the sign of β parameter. Specifically, λ is increasing when $\beta > 0$ and decreasing when $\beta < 0$. Of course, it becomes a constant function of time when β is zero.

Hence, our set of possible statistical representations allows to capture a wide range of (deterministic) time varying intensity processes. Considered failure rates can be whether monotonous, hump-shaped, convex or concave. Nevertheless, the adequacy of our selected representations depends on the observed and empirical patterns describing the corporate failures under consideration.

3 Estimation and adequacy

Having a set of potential distributions aimed at characterizing our firms' lifetimes, we are going to estimate the related parameters. For this purpose, we present briefly the data we use. Then, we introduce our estimation method, and end with an adequacy test in order to check for the consistency of the chosen distributions.

3.1 Parameter estimation

Using empirical default probabilities, we estimate each distribution's parameters while minimizing the cumulative squared error relative to the corresponding set of parameters. When such parameters are both finite and in adequacy with our theoretical framework, we perform a Kolmogorov-Smirnov test to investigate the coherency of our theoretical probability distributions.

3.1.1 Data

We study French firm's bankruptcy using monthly empirical default probabilities $\hat{p}_t$ of Gatfaoui (2003) ranging from january 1990 to december 1999, namely time ranges from 1 month to 120 months. Such probabilities are computed as the ratio of the number of defaulting firms during month t to the number of listed existing firms during this month.[11] We consider the empirical aggregate default probabilities[12] of 16 economic sectors, which correspond to motor trade and repairing industry (AU), consumer goods (BC), capital goods (BE), intermediate goods and energy (BI), construction and civil engineering (BP), specialized food retail trade (DA), non-specialized retail trade (DN), other specialized retail trade (DS), food wholesale trade (GA), non-food wholesale trade (GN), hotels, catering and cafés industry (HR), food processing sector (IA), real estate (IM), business services (SE), private services (SP), and finally transport industry (TT). 'TOTAL' refers to the empirical default probability all sectors included, or equivalently, the global default probability. These default probabilities are shown to be asymmetric and generally exhibit a negative excess of kurtosis.

[11] We do not compute hazard rates since the sample of existing firms during our time horizon incorporates newly created firms.

[12] We compute default probabilities that are aggregated at each economic sector's level (i.e., all ratings included). This way, we obtain a general trend for corporate defaults while observing failures among each sector.

Recall that our framework assumes that we observe $\hat{p}_t = \hat{F}(t)$ for each time $t \in \{1, ..., 120\}$. Our work is now to try to fit our theoretical distributions to the empirical behavior of French firms' lifetimes.

3.1.2 Estimation method and results

Our aim is to attempt to characterize soundly French firms' lifetimes. For this purpose, we are going to try to fit a set of theoretical probability distributions to our French empirical default probabilities. Let C be the set of our eight potential probability distributions. Each statistical law depends on a set of parameters $\theta \in \Theta$. For example, $\theta = \{\mu, \sigma\}$ and $\Theta = \mathbb{R} \times \mathbb{R}_+^*$ for the lognormal distribution belonging to C. Let $F_\theta(t)$ and $\lambda_\theta(t)$ be the corresponding theoretical cumulative distribution function and failure rate respectively.

To estimate the set of parameters $\hat{\theta}$ fitting our empirical default probabilities, we solve a quadratic minimization problem. We realize the minimization of the sum of squared observed errors as follows:

$$\hat{\theta} = \arg\min_{\theta} \left\{ \sum_{t=1}^{120} \left(\hat{F}(t) - F_\theta(t) \right)^2 \right\} \tag{33}$$

The resolution of this quadratic problem is achieved while using the Polak-Ribiere Conjugate Gradient methodology.[13] We present therein the conclusions of our estimation method. To spare space, we do not display the related results when they are inconsistent since these ones are not interesting for the rest of the paper.

Concerning the lognormal distribution, our minimization algorithm does not converge for finite values of $\hat{\mu}$ and $\hat{\sigma}$. Therefore, this distribution is incompatible with our theoretical framework. In the log-logistic case, the minimization converges towards finite values of parameters for only five sectors, namely SP, HR, SE, IM and GN. Concerning the other economic sectors, the algorithm does not converge towards finite values of $\hat{\mu}$ and $\hat{\sigma}$. And, our corresponding results are displayed in the table underneath.

Table 1: Log-logistic parameters

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
$\hat{\mu}$	68.2402	125.5136	29.3724	106.0047	77.2463
$\hat{\sigma}$	19.4402	34.9037	8.0572	24.9806	17.2062

Since the estimated shape parameter $\hat{\sigma}$ lies above unity, the log-logistic distribution implies a strong convex decreasing behavior for the failure rate in accordance with Gatfaoui (2003).

[13] Refer to Polak (1971) and Press *et al.* (1992) among others for more details about this optimization method.

The gamma law's estimation is such that the algorithm converges towards a positive scale parameter but a negative shape parameter for each sector, which is inconsistent with our theoretical framework assuming a positive shape parameter. Indeed, a negative shape parameter implies both a negative theoretical expectation[14] for the lifetime and a negative theoretical variance, which is incoherent. Thus, the two first moments are not defined in such a setting. In the same way, the weibull's estimation leads to a negative shape parameter for all sectors and also a negative location parameter for some of them. Such results are contrary to the assumptions underlying this distribution. Therefore, the weibull distribution is inconsistent with the general behavior of French firm's lifetimes. Analogously, the second species beta distribution is not suitable for modeling the lifetimes of French firms. Indeed, the results exhibit a positive value of $\hat{p}$ but a negative value of $\hat{q}$ (i.e., contrary to theoretical assumption) for all the sectors under consideration.

The estimations corresponding to the mixture of exponential laws are not far from the previous conclusion. Only SP sector can be represented by such a distribution. The other sectors exhibit generally a negative $\hat{\lambda}_1$ parameter and a positive $\hat{\lambda}_2$ parameter or the reverse in some cases. Those results are incompatible with the theoretical framework. We then find:

Table 2: Parameters for mixture of exponential laws

Sector	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
SP	0.9870	0.0130	1.3976e-005	5.7764

The empirical probability that SP sector's lifetime follows an exponential law of the first type (i.e., an exponential law with intensity $\hat{\lambda}_1$) is extremely high. We also performed the corresponding estimations for a mixture of exponential laws with equiprobability (i.e., $\pi_1 = \pi_2 = \frac{1}{2}$). Estimation results show that whether $\hat{\lambda}_1$ or $\hat{\lambda}_2$ is negative which is inconvenient. Therefore, a mixture of exponential laws does not seem adapted to describe French bankruptcies.

On the contrary, the Cox-Lewis distribution's estimation leads to coherent results for all economic sectors, indicating that such a statistical law is appropriate to describe French bankruptcies. The related estimations are given in the table underneath.

Table 3: Cox-Lewis parameters

[14] Recall that a lifetime is positive.

Sector	AU	BC	BE	BI	BP	DA
$\hat{\alpha}$	-2.4440	-2.0748	-3.0046	-2.0627	-1.9593	-2.2075
$\hat{\beta}$	-3.1448	-3.0248	-1.3567	-4.0070	-3.8640	-4.9615
Sector	**DN**	**DS**	**GA**	**GN**	**HR**	**IA**
$\hat{\alpha}$	-1.9346	-2.4457	-2.0959	-4.2696	-3.0655	-2.4848
$\hat{\beta}$	-5.2897	-3.8166	-3.8987	-0.3850	-1.5432	-3.9962
Sector	**IM**	**SE**	**SP**	**Total**	**TT**	
$\hat{\alpha}$	-5.4703	-2.5166	-2.8646	-2.4751	-2.2966	
$\hat{\beta}$	-0.0938	-4.8638	-4.0902	-3.1545	-3.7812	

Since all the shape parameters $\hat{\beta}$ are negative, we can conclude that from january 1990 to december 1999, French firms live healthy times in general since their credit quality improves. Namely, French failure rates are convex decreasing functions of time. This feature is mostly due to the good side of the business cycle in France during this time period. Moreover, $\hat{\beta}$ is clearly different from zero in most cases except for GN and IM sectors.[15] Broadly speaking, such a behavior is contrary to the classical exponential law assumption of Gatfaoui (2003).

In a less powerful way, the exponential exponent law's representation matches only 5 economic sectors such as GN, HR, IM, SE and SP. Concerning the rest of the studied economic sectors, our minimization algorithm converges towards negative values of the shape parameter $\hat{b}$, which is inconsistent with our theoretical setting. We display the corresponding results in the table below.

Table 4: Exponential exponent parameters

Sector	GN	HR	IM	SE	SP
$\hat{a}$	0.0295	0.0271	0.0258	0.0140	0.0112
$\hat{b}$	0.0505	0.0283	0.1213	0.0446	0.0580

Since $\hat{b}$ lies below unity, French firms' failure rates are convex decreasing functions of time. Moreover, the estimated shape parameter lies far from unity, which is again contrary to the classical exponential law assumption. We are going to check for this feature in the next subsection.

3.2 Adequacy test

We process in two steps here. First, considering our non-homogeneous Poisson processes, we investigate whether the estimated parameters exhibit a classical exponential law behavior for our failure rates. To this end, we achieve an exponentiality test. Second, we check for the adequacy of our convenient

[15] We are going to investigate later such a result for those sectors.

statistical representations with French empirical default probabilities. We resort to Kolmogorov-Smirnov test for this purpose.

The exponentiality test is aimed at testing whether our non-homogeneous Poisson processes correspond to a classical exponential law with constant intensity or not. This is equivalent to test whether $\beta = 0$ for the Cox-Lewis process or $b = 1$ for the exponential exponent process. Concerning the Cox-Lewis process case, we assume that $\beta \neq 0$. Given the complex nature of the Cox-Lewis process, an exponentiality test cannot be achieved easily and requires a more advanced analysis,[16] which is not the goal here. One straightforward test would be to compare the results we get when $\beta \neq 0$ (i.e., a pure Cox-Lewis process) with the ones we obtain when $\beta = 0$ (i.e., classical exponential law as stated in Gatfaoui [2003]). Therefore, our exponentiality test will be summarized in the next section while looking for an optimal representation of our empirical default probabilities, in terms of best fitting. Incidentally, let us underline that the inadequacy of the time independence assumption underlying the classical exponential model as applied to credit risk valuation supports a Cox-Lewis modeling with $\beta \neq 0$.

The theoretical framework stated by relation (31) for the exponential exponent case implies that:

$$Y(t) = \rho \ln(t) \tag{34}$$

with

$$Y(t) = \ln\left[-\frac{\ln(1-p_t)}{a}\right] - \ln(t) \tag{35}$$

$$\rho = b - 1 \tag{36}$$

The exponentiality test related to the second non-homogeneous Poisson process is then equivalent to test whether $\rho = 0$ or not. We realize therefore the regression of $\hat{Y}(t) = \ln\left[-\frac{\ln(1-\hat{p}_t)}{\hat{a}}\right] - \ln(t)$ on $\ln(t)$, for our five sectors, to achieve such a test. Our first results show a positive first order autocorrelation of the related regressions' residuals while considering the corresponding Durbin Watson's statistic[17] (which is always below unity). For reasons of parsimony, we do not report these non-interesting results here. To bypass the autocorrelation problem, we process to the minimization of the following sum of squared

[16] Indeed, this kind of test is complex in our case insofar as we do not observe X_t but only $F(t)$ for each t in $\{1, ..., 120\}$.

[17] We also performed a Phillips-Perron unit root test, which showed that $\hat{Y}(t)$ was a first order-integrated series for each of the five studied sectors.

errors:[18]

$$\min_{\rho}\left\{\sum_{t=1}^{120}\left(\hat{Y}\left(t\right)-\rho\ln\left(t\right)\right)^{2}\right\} \tag{37}$$

which gives the following results:[19]

Table 5: Exponential exponent's regression parameters

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
$\hat{\rho}$	-0.9443*	-0.9745*	-0.8855*	-0.9576*	-0.9516*
Skewness	1.1724	1.2526	1.1352	0.6264	1.4573
Kurtosis	4.5636	5.0911	4.5513	4.0145	5.9134
Jarque-Bera	39.7157	53.2454	37.8079	12.9947	84.9146

* Significant at a 1% level of Student t-test.

We see that $\hat{\rho}$ is significantly different from zero (i.e., $\hat{b}$ is significantly different from unity), which confirms the non-classical exponential law assumption. We also give some descriptive statistics related to $\hat{Y}$, and which show the non-normality of $\hat{Y}$.

Let us now introduce the obtained results for the Kolmogorov-Smirnov adequacy test. This test is aimed at assessing the appropriateness of theoretical distributions to the empirical observed behaviors (i.e., empirical distributions). We display the corresponding empirical adequacy statistic in the tables underneath for each convenient probability representation.

Table 6: Log-logistic Kolmogorov statistic

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
Statistic	0.1250	0.1418	0.2617	0.0751	0.0537

Table 7: Kolmogorov statistic for mixed exponential laws

[18] This is equivalent to maximize the corresponding log-likelihood function while assuming normal regression residuals, or to employ the generalized method of moments (i.e., GMM) with three moment conditions here (i.e., the zero expectation assumption for residuals, the constant variance of residuals, and the zero cross-correlations assumptions between residuals). Moreover, testing for overidentifying conditions with Hansen's (1982) J-statistic in GMM estimation, we accept the H_0 orthogonality assumption (i.e., no unsatisfied overidentifying restriction or, equivalenlty, no violation of extra moment restrictions) at a 1% level for our five sectors GN, HR, IM, SE and SP. Refer to Hamilton (1994), Mittelhammer *et al.* (2000) or Ruud (2000), among others, for further explanations about Hansen's (1982) test and overidentifying restrictions.

[19] Notice that we have the following bounded absolute difference $0.0021<\left|\hat{\rho}-\left(\hat{b}-1\right)\right|<0.0068$, which underlines the soundness of our previous estimation and also supports the non-classical exponential distribution assumption.

Sector	Statistic
SP	0.0554

Table 8: Cox-Lewis Kolmogorov statistic

Sector	**AU**	**BC**	**BE**	**BI**	**BP**	**DA**
Statistic	0.1004	0.1873	0.2873	0.2336	0.1598	0.1020
Sector	**DN**	**DS**	**GA**	**GN**	**HR**	**IA**
Statistic	0.1625	0.0966	0.2001	0.1305	0.1341	0.1005
Sector	**IM**	**SE**	**SP**	**Total**	**TT**	
Statistic	0.2354	0.0740	0.0555	0.1006	0.1236	

Table 9: Exponential exponent's Kolmogorov statistic

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
Statistic	0.1250	0.1418	0.2618	0.0753	0.0537

Given a five percent level of test, the corresponding critical value of the Kolmogorov-Smirnov statistic is approximatively 1.3412 for a number of observations equal to 120. Our results show that we accept the null hypothesis of adequacy for our convenient probability distributions. Namely, consistent estimated parameters of distributions match conveniently the empirical observed behavior of French failures. The following of our work consists therefore of choosing the optimal probability representation compared to the classical exponential law of Gatfaoui (2003), and to induce the related implications.

4 Optimal selection and default probabilities

Given the convenient possible statistical representations of French failures, we currently face the task of selecting the optimal distribution. Once it is determined, the optimal representation allows to compute the corresponding forward conditional default probabilities.

4.1 Optimal distribution's choice

We are facing a selection problem concerning the choice of the most appropriate distribution that describes French bankruptcies. For this purpose, we choose the average absolute error as a selection criterion. Therefore, the most realistic representation fitting our empirical French default probabilities corresponds to the theoretical probability distribution, which minimizes the average absolute error[20] as follows:

$$F^*_{\hat{\theta}} = \min_{F_\theta \in C} \left\{ \frac{1}{120} \sum_{t=1}^{120} \left| \hat{F}(t) - F_{\hat{\theta}}(t) \right| \right\} \tag{38}$$

where $\hat{F}$ is the empirical cumulative distribution function and $F_{\hat{\theta}}$ is the theoretical cumulative distribution function employed with $\hat{\theta}$ parameter estimates. We display in the tables underneath the results[21] we get relative to the average absolute error for each convenient representation in C set.

Table 10: Log-logistic average absolute error

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
Statistic	0.00389083	0.00345087	0.008771	0.001942	0.00149397

Table 11: Average absolute error for mixed exponential laws

Sector	**Statistic**
SP	0.001554

Table 12: Cox-Lewis average absolute error

Sector	AU	BC	BE	BI	BP	DA
Statistic	0.002815	0.005809	0.007836	0.007033	0.003810	0.002588
Sector	DN	DS	GA	GN	HR	IA
Statistic	0.005075	0.002813	0.003953	0.00387747	0.00356350	0.002709
Sector	IM	SE	SP	Total	TT	
Statistic	0.007328	0.002017	0.001639	0.002802	0.003366	

Table 13: Exponential exponent's average absolute error

Sector	**GN**	**HR**	**IM**	**SE**	**SP**
Statistic	0.00389129	0.00345071	0.008775	0.001939	0.00149367

[20] Specifically, the most appropriate representation minimizes the L^1-norm distance. We could also have used the L^2-norm distance, which corresponds to the square root of the sum of squared error. These two methodologies are equivalent and give the same results in terms of optimal selection.

[21] We had to add generally two more decimals to allow an easier comparison between possible representations, and four more decimals for GN, HR and SP sectors' representations to better discriminate between results.

We also display therein the average absolute error obtained in Gatfaoui (2003) while using the classical exponential representation.

Table 14: Classical exponential's average absolute error

Sector	AU	BC	BE	BI	BP	DA
Statistic	0.012148	0.021474	0.020258	0.017890	0.017198	0.009677
Sector	DN	DS	GA	GN	HR	IA
Statistic	0.015079	0.011391	0.015517	0.01532811	0.0135336	0.010531
Sector	IM	SE	SP	Total	TT	
Statistic	0.019761	0.007418	0.005748	0.012562	0.012865	

Given our selection criterion (38), we find that AU, BC, BE, BI, BP, DA, DN, DS, GA, GN, IA, IM, Total and TT sectors are optimally described by a Cox-Lewis process whereas HR, SE and SP sectors are optimally represented by an exponential exponent distribution. Notice that such an optimal selection criterion also constitutes an exponentiality test highlighting the non-classical exponential feature of French default probabilities. Indeed, non-homogeneous Poisson processes seem more appropriate to describe French bankruptcies, and currently underline a convex decreasing behavior of corresponding hazard rates relative to time in the lens of their respective parameter estimates. Such a feature is also supported by TOTAL's behavior, which gives the general trend of French failures here in accordance with the convex decreasing implied hazard rates exhibited by Gatfaoui (2003).

4.2 Forward conditional default probabilities

Having knowledge about the optimal characterization of French failures, we are able to achieve some forecasts concerning French bankruptcies. More precisely, we are able to compute the monthly probability that a given firm defaults in the n forthcoming months provided that it has not defaulted before time $t \in \{1, ..., 120\}$, or equivalently, january 1990-december 1999 (i.e., the range of our observed sample period). The corresponding conditional forward default probability then writes on the basis of optimal representations $F^*_{\hat{\theta}}$:[22]

$$P\left(X < t+n \mid X > t\right) = \frac{P\left(t < X < t+n\right)}{P\left(X > t\right)} = \frac{F^*_{\hat{\theta}}\left(t+n\right) - F^*_{\hat{\theta}}\left(t\right)}{1 - F^*_{\hat{\theta}}\left(t\right)} \tag{39}$$

For all the sectors under consideration, we compute the related conditional forward default probabilities on the forthcoming one year, two years and five years horizons for $t = 120$ (i.e., forecasts of forward conditional default probabilities).

[22] The forward conditional survival probability can also be obtained by computing the symmetric quantity $P\left(X > t+n \mid X > t\right) = 1 - P\left(X < t+n \mid X > t\right) = \frac{1-F^*_{\hat{\theta}}(t+n)}{1-F^*_{\hat{\theta}}(t)}$.

While realizing our estimations, we assume implicitly that the business cycle's trend will remain stable over the n forthcoming months following january 1999 (i.e., some favorable scenario). Our related results are displayed in the table underneath:

Table 15: Forward conditional default probabilities (in percent)

Sector	1 year	2 years	5 years
AU	0	0	0
BC	0	0	0
BE	0	0	0
BI	0	0	0
BP	0.000039	0.000052	0.000058
DA	0	0	0
DN	0	0	0
DS	0	0	0
GA	0	0	0
GN	0	0	0
HR	0.008368	0.016026	0.035750
IA	0	0	0
IM	0	0	0
SE	0.007382	0.014148	0.031619
SP	0.008160	0.015648	0.035023
TT	0	0	0
Total	0	0	0

Whatever the chosen forthcoming horizon following the end of our sample period, the forward conditional default probabilities we generally get are so small that they can be set to zero in value. This feature is not surprising given the strong convex and fast decreasing behavior of the corresponding failure rates. Indeed, French failure rates are strongly decreasing since the corresponding estimated parameters are negative and the time horizon is long (i.e., high time value in months). Hence, forward conditional default probabilities are generally stable whatever the coming time window under consideration. By the way, the level of failure rates is quasi-zero for most of economic sectors except for BP, HR, SE and SP sectors. The later sectors exhibit a conditional forward default probability as high as the forthcoming horizon is long. Such features rely on possible and plausible economic explanations. During our studied decade, BP sector experiences both a real estate crisis and a restructuring of the construction branch. Differently, the number of annual failures for SE sector increases slightly (i.e., a two percent average increase over ten years). As regards HR and SP sectors, the high probability levels may result from the non-negligible number of business start-ups given that the mortality rate of young firms is extremely high during their first five years of existence. Moreover, compared to the results obtained by Gatfaoui (2003), the classical exponential model tends to greatly overestimate forward conditional default probabilities. Moreover, this low general forward default risk behavior is due to the global business cycle's growth characterizing our sample period. In addition, we also display in the appendix the forward conditional default probabilities we get for the same time horizons, but starting from the beginning of our sample period, namely current time $t = 1$, or equivalently, on january 1990. As expected, in-sample forward conditional default probabilities (i.e., estimated at time $t = 1$) are higher than out-sample forward conditional default probabilities (i.e., estimated at time $t = 120$). Indeed, estimating forward conditional default probabilities at the far end of our time horizon assumes that the trend of the business cycle's growth will remain stable over the coming time horizon under consideration.

5 Credit spreads

We first present our theoretical framework that allows for assessing the value of any credit risky discount bond. Then, we apply our setting to compute the related credit spreads.

5.1 Discount bonds

We introduce here the link between reliability and the reduced form approach of credit risk while valuing discount bonds and related credit spreads. First, we set our basic assumptions and valuation framework. Then, we deduce the credit spread's term structure implied by credit risky discount bonds.

Reliability attempts to determine the time when a given firm may end its life, or equivalently, its survival time. Any firm's lifetime is assumed to stop when the firm under consideration defaults. Therefore, any firm's default probability is also linked to the arrival time of its potential default. Let t_d be the default time, or equivalently, the (first) date when a default event occurs. Given our framework, t_d could be defined as the first time (between the issuing of the firm's debt $t = 0$ and its corresponding maturity T) when the firm's value crosses down a fixed critical threshold known as its default barrier.[23] In such a framework, t_d is a random variable also called default stopping time, and satisfies the following relations in the light of reliability:[24]

$$F(t) = P(X < t) = P(t_d < t) \tag{40}$$

$$R(t) = P(X > t) = P(t_d > t) \tag{41}$$

Hence, $P(t_d < t)$ represents the probability of transition at time t from a non-default state to a default state whereas $P(t_d > t)$ represents the probability

[23] In practice, the default point lies between the firm's current short term debt and its current total debt (i.e., the sum of its current long term and short term debt).

[24] The law describing the firm's lifetime is the same as the probability distribution describing the default time variable.

of transition from a non-default state to a non-default state (i.e., stability of the firm's working state). Recall that default consists of an absorbing state, which implies that any firm remains definitively in a failure state once it has defaulted.[25] Consequently, our framework implies the following conditional probabilities:

$$P\left(X_t > X_{t_d} \mid t_d < t\right) = 0 \tag{42}$$

$$P\left(X_t < X_{t_d} \mid t_d < t\right) = 1 \tag{43}$$

In the lens of reduced form approach, we are able to price credit risky bonds under some regularity assumptions. First, consider a credit risky discount bond $B_d\left(t,T\right)$ at current time t with maturity T, and assume that this discount bond allows for a fixed payment only at maturity.[26] The payment received by the discount bond's holder is conditional on the occurrence of a default event before maturity T. Indeed, the payment corresponds to a unit of currency if no default event has occurred before T whereas it corresponds to a constant partial amount δ of currency if a default event occurs before maturity T. Namely, the bond's corresponding payment at expiration is:

$$B_d\left(T,T\right) = \begin{cases} 1 & \text{if} \quad t_d > T \\ \delta & \text{if} \quad t_d \leq T \end{cases} \tag{44}$$

Notice that $\delta \in [0,1]$ corresponds to the recovery rate[27] received by the debtholders of the issuing firm under consideration in case of a default occurring before maturity.[28] Moreover, the final payment provided by the risky discount bond at maturity can be written as $B_d\left(T,T\right) = \mathbf{1}_{\{t_d>T\}} + \delta\mathbf{1}_{\{t_d\leq T\}}$ where $\mathbf{1}_{\{A\}}$ is equal to 1 if $\{A\}$ is satisfied and 0 else. Second, consider a risk free discount bond $B\left(t,T\right)$ at current time t with maturity T, related to a deterministic risk free interest rate $r\left(t\right)$ at most. Consequently, the risk free discount factor satisfies the following relation whatever $0 \leq t \leq T$:

$$B\left(t,T\right) = \exp\left\{-\int_t^T r\left(s\right)ds\right\} \tag{45}$$

[25] It is impossible for any defaulted firm to recover solvency (i.e., equation [42]), and therefore, to start working again in a reliability sense after a default event (i.e., equation [43]). Namely, equation [42] represents the probability of transition at time t from a default state to a non-default state while equation [43] represents the probability of transition from a default state to a default state (i.e., stability of the firm's non-working state).

[26] Such a setting is aimed at representing firms issuing homogeneous debts.

[27] In practice, the recovery rate is different from unity. We could rewrite the final discount bond's price in a more general form such that $B_d\left(T,T\right) = \delta$ where $\delta = 1$ when $t_d > T$, and $\delta \in [0,1[$ when $t_d \leq T$.

[28] The recovery rate is paid to debtholders at maturity when the default barrier is reached before the discount bond's expiration date.

Given our assumptions and framework, we are able to price the risky discount bond in the universe endowed with probability[29] P as the discount value of its final payment:

$$B_d(t,T) = \mathbb{E}^P\left[B(t,T)\, B_d(T,T) \mid t_d > T\right] \tag{46}$$

where $\mathbb{E}^P[.]$ is the unconditional expectation operator relative to the probability measure P, such that we finally get:

$$B_d(t,T) = B(t,T)\left[\frac{1-F(T)}{1-F(t)} + \delta\frac{F(T)-F(t)}{1-F(t)}\right] \tag{47}$$

Given relation (3), the former expression becomes:

$$B_d(t,T) = B(t,T)\left[(1-\delta)\ R_t^T + \delta\right] \tag{48}$$

with

$$R_t^T = \exp\left\{-\Lambda_t^T\right\} = \exp\left\{-\int_t^T \lambda(s)\, ds\right\} = R(T) - R(t) \tag{49}$$

where $\Lambda_t^T = \displaystyle\int_t^T \lambda(s)\, ds$ is the cumulative hazard rate between t and T. Notice that given definitions (29) and (32), we have then for the Cox-Lewis and exponential exponent processes respectively:[30]

$$\Lambda_t^T = \frac{e^{\alpha}}{\beta}\left(e^{\beta T} - e^{\beta t}\right) \tag{50}$$

$$\Lambda_t^T = a\left(T^b - t^b\right) \tag{51}$$

Thus, we are able to value credit risky discount bonds while knowing only their respective hazard rate functions and the risk free term structure (see, for example, Jeanblanc & Rutkowski [2002] for more details and explanations).

To go further, we introduce the respective yields to maturity $Y(t,T)$ and $Y_d(t,T)$ corresponding to the risk free term structure and the credit risky discount bonds as follows for each $t \in [0,T]$:

$$B(t,T) = \exp\left\{-Y(t,T)\ (T-t)\right\}\ B(T,T) \tag{52}$$

[29] We assume that P is the pricing measure inferred from market data. All the regularity conditions ensuring that P is a measure equivalent to the historical (i.e., original) one P^h such that risky assets' discount prices are P-martingales, are assumed to hold here. Under both the incomplete market and the arbitrage-free principle assumptions, let $\mathcal{Q}$ be the set of martingale measures equivalent to P^h. We therefore know that any risky bond's price lies in the following arbitrage-free prices' bracket $B_d^P(t,T) \in \left[\inf_{Q\in\mathcal{Q}} B_d^Q(t,T),\ \sup_{Q\in\mathcal{Q}} B_d^Q(t,T)\right]$ (see Giesecke & Goldberg [2003] for example). Notice that we could also assume that both P is the historical probability and investors are risk-neutral.

[30] Here, the Cox-Lewis shape parameter β is assumed to be non-zero.

$$B_d(t,T) = \exp\{-Y_d(t,T)\ (T-t)\}\ B_d(T,T) \quad (53)$$

Hence, the related credit spread takes the following form:

$$S(t,T,\delta) = Y_d(t,T) - Y(t,T) = -\frac{1}{T-t}\ln\left\{\frac{B_d(t,T)}{B(t,T)}\right\} \quad (54)$$

which leads to the new expression:

$$S(t,T,\delta) = -\frac{1}{T-t}\ln\left\{(1-\delta)\ \exp\left\{-\Lambda_t^T\right\} + \delta\right\} \quad (55)$$

Thus, our framework allows us to describe and compute credit spreads while knowing only the hazard rate functions of the credit risky discount bonds under consideration (see Fons [1994] among others). Recall that such default rate functions are obtained from the empirical behavior of monthly aggregate default probabilities. Therefore, we compute analogously monthly aggregate credit spreads among sectors (i.e., all ratings included for a given sector).

5.2 Estimations

All the assumptions stated in the previous subsection are assumed to hold here. Since we have already determined the optimal representations of French failures along with reliability and therefore the corresponding hazard rate functions, we are going to apply the previous framework to value the related theoretical credit spreads' levels.

Recall that HR, SE and SP sectors' lifetimes follow an exponential exponent distribution process, which implies that the theoretical credit spread related to our credit risky discount bond's setting expresses:

$$S(t,T,\delta) = -\frac{1}{T-t}\ln\left\{(1-\delta)\ \exp\left\{-a\ \left(T^b - t^b\right)\right\} + \delta\right\} \quad (56)$$

Given that the fourteen other sectors' lifetimes follow a Cox-Lewis process, the corresponding theoretical credit spreads then write:

$$S(t,T,\delta) = -\frac{1}{T-t}\ln\left\{(1-\delta)\ \exp\left\{-\frac{e^\alpha}{\beta}\left(e^{\beta T} - e^{\beta t}\right)\right\} + \delta\right\} \quad (57)$$

Such a characterization allows us to compute the related term structures of credit spreads at any given time t. Due to the strong convex and fast decreasing behavior of French failure rates over our time sample, we choose to estimate the theoretical term structure of related credit spreads at current time $t = 1$ (i.e., on january 1990), for time horizons (i.e., time to maturity $T - t$) corresponding to 1, 2 , 5 and 10 year(s) successively. Moreover, the recovery rate is allowed to take two distinct values, namely zero (i.e., total loss for debtholders) or 50%

(i.e., medium loss scenario).[31] We display our results in the tables below for each value taken by the recovery rate δ. We study two distinct cases, namely a zero recovery rate situation and a 50% recovery rate setting.

Table 16: Theoretical credit spreads for $\delta = 0$ (in basis points)

Sector	**1 year**	**2 years**	**5 years**	**10 years**
AU	0.1944	0.0972	0.0389	0.0194
BC	0.3195	0.1598	0.0639	0.0320
BE	5.3811	2.6906	1.0762	0.5381
BI	0.1068	0.0534	0.0214	0.0107
BP	23.0054	15.2351	6.7867	3.4056
DA	0.5054	0.2527	0.1011	0.0505
DN	0.4163	0.2082	0.0833	0.0416
DS	0.1293	0.0647	0.0259	0.0129
GA	0.1148	0.0574	0.0230	0.0115
GN	20.3992	10.3001	4.1205	2.0602
HR	1.6976	1.0752	0.5564	0.3278
IA	0.6379	0.3189	0.1276	0.0638
IM	0.4808	0.2404	0.0962	0.0481
SE	1.4136	0.9003	0.4694	0.2782
SP	1.4907	0.9538	0.5005	0.2980
TT	0.9908	0.4954	0.1982	0.0991
Total	0.9484	0.4742	0.1897	0.0948

Drawing the same conclusions whether the recovery rate is zero or 50%, we give our general comments about credit spreads' levels after the next table.

Table 17: Theoretical credit spreads for $\delta = 0.5$ (in basis points)

[31] When the recovery rate is zero, we then have $S(t,T,0) = \frac{1}{T-t}\int_t^T \lambda(s)\, ds$. The theoretical credit spread corresponds to the average cumulative hazard rate on the remaining time to maturity of debt.

Sector	1 year	2 years	5 years	10 years
AU	0.0972	0.0486	0.0194	0.0097
BC	0.1598	0.0799	0.0320	0.0160
BE	2.6862	1.3431	0.5372	0.2686
BI	0.0534	0.0267	0.0107	0.0053
BP	11.4233	7.5479	3.3588	1.6854
DA	0.2527	0.1263	0.0505	0.0253
DN	0.2081	0.1041	0.0416	0.0208
DS	0.0647	0.0323	0.0129	0.0065
GA	0.0574	0.0287	0.0115	0.0057
GN	10.1372	5.1182	2.0475	1.0237
HR	0.8484	0.5373	0.2780	0.1637
IA	0.3189	0.1594	0.0638	0.0319
IM	0.2404	0.1202	0.0481	0.0240
SE	0.7065	0.4499	0.2345	0.1390
SP	0.7450	0.4766	0.2500	0.1489
TT	0.4953	0.2476	0.0991	0.0495
Total	0.4741	0.2370	0.0948	0.0474

Whatever the value of the potential recovery rate, the three highest credit spreads by descending order concern BP, GN and BE sectors respectively. The smallest computed credit spreads relate to BI sector. Moreover, the credit spreads estimated for TOTAL sector give general and average trends for French credit spreads under our basic framework and assumptions. Incidentally, BE, BP, GN, HR, SE and SP sectors exhibit higher credit spreads than those estimated for TOTAL sector, the eleven remaining French sectors exhibiting smaller credit spreads' levels. On average, credit spreads decrease by 50.0599% when switching from a zero recovery to a 50% recovery scenario[32] (all time horizons included). Incidentally, the obtained theoretical sector aggregate credit spreads are similar to the credit spreads' levels computed for AAA and AA rating classes according to the standard of Moody's rating agency (at the beginning of the 90' economic growth, or equivalently, the favorable prevailing business cycle).

As a rough guide, we also plot the theoretical term structure of our credit spreads for TOTAL sector as a function of both time to maturity and recovery rate. Thus, we get a general trend for the theoretical credit spreads' term structure related to our French bankruptcies on january 1990 (i.e., at current time $t = 1$ as represented in Fig. 1 below).

As expected, TOTAL sector's credit spreads are as high as the recovery's level is low. Moreover, the shorter the debt's time to maturity, the higher those credit spreads are. Of course, these credit spreads are zero when the recovery is 100% since there is no risk of loss for debtholders. In such a case, any discount

[32] The percentage of reduction remains slightly the same whatever the time horizon under consideration.

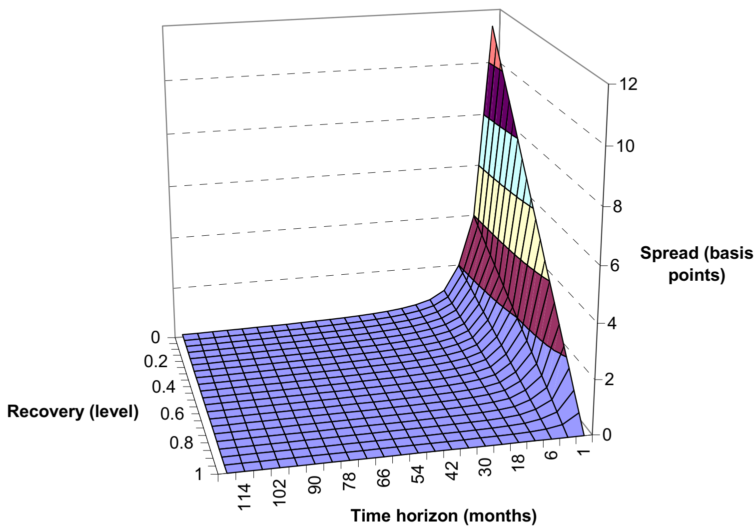

Figure 1: Theoretical credit spread's term structure for TOTAL sector in january 1990.

bond's investor becomes certain to receive the risky discount bond's final unit payment. Such levels and features for credit spreads are highly explained by the global good side of the business cycle in France during the 1990-1999 decade. Indeed, Fisher (1959) shows the impact of business cycle on credit spreads. Specifically, these global risk premia fluctuate through time with a specific behavior. Credit spreads tend to increase during crisis time period whereas they tend to decrease during economic growth.

6 Conclusion

In this paper, we have tried to apply some quantitative tools for a credit risk management purpose. Since credit risk encompasses the possibility of social, economic and financial harms, some control setting and some credit risk management policies have to be determined in order to minimize the harmful effects of disastrous risky events such as failures. Such a process requires to define and quantify the combinations of events that are likely to trigger a bankruptcy, namely our top-event. Fault tree theory, which is an alternative approach of reliability, consists of such a process as far as any disastrous event is defined by both its frequency and its consequences. Such a characterization is aimed at helping to prevent the occurrence of the top-event, or equivalently, bankruptcy.

Along with this point of view, we have extended the framework of Gatfaoui (2003) to assess default risk while employing the fault tree approach. This author describes French firms' lifetimes and failure rates while using a classical exponential law with constant intensity. We have proposed a set of eight probabilistic representations for failure rates, which encompass sometimes the classical exponential law as a special case. We have found that French bankruptcies are better described by a non-homogeneous Poisson process, and therefore a time varying failure rate, such as Cox-Lewis or exponential exponent-type distribution. The obtained French failure rates are convex decreasing functions of time. Indeed, fourteen sectors are optimally represented by a Cox-Lewis process with a negative shape parameter whereas the three remaining sectors are optimally described by an exponential exponent process with a shape parameter lying far below unity.

Once our French failures are optimally characterized, we compute the corresponding one, two and five year(s) forward conditional default probabilities. The results we get show that conditional default probabilities are zero for most economic sectors expected for BP, HR, SE and SP sectors. These findings come from the strong decreasing behavior of related hazard rates. Specifically, French sectors' hazard rates generally tend to be zero at the end of our sample period, implying their zero value for even longer time horizons. We have also noticed the misestimations induced by the classical exponential law. In particular, using a classical exponential law to model French failure rates generates a strong overestimation bias while assessing forward conditional default probabilities. Consequently, a sound method of valuation of French failures requires the use Cox-type processes. However, our estimations are achieved on a time interval encompassing a favorable business cycle (and assuming thus the same economic trend in the future). The occurrence of any business cycle's reversal just after our sample period would lead to biased forward estimations of default since contrary to the future trend. Therefore, we have to take into account and to realize expectations about forthcoming business cycles in order to assess soundly default risk. Indeed, any credit risk valuation method should encompass the future business cycle's trend in a forecasting prospect. Moreover, firms' lifetimes and then related failure rates are known to depend strongly on time varying explanatory variables (see Altman [1993] for example). Thus, one way to solve this bias problem would be to employ a Cox-type process with a time varying intensity parameter depending on both accounting, financial and above all macroeconomic variables in order to account for the business cycle's effect. This suggests two straightforward possible extensions in order to achieve realistic failure forecasts encompassing business cycle's reversal in a dynamic framework.[33]

First, we could apply a more complex approach of fault tree requiring stochastic processes to assess probabilities of transition from one state to another. Such a process would employ Cox-type processes with stochastic intensities. Specifically, the intensity, or equivalently, the failure rate could depend on stochastic variables such as firm value and/or its solvency ratio (to encompass financial and accounting information), as well as interest rates among others

[33] We give an example of dynamic fault tree's application in the appendix, provided that the appropriate assumptions and default framework are stated.

(to account for business cycle's effect). Although explored by the reduced form approach of credit risk (see Gill & Johansen[34] [1990], Lando[35] [1998] or Jarrow & Yu[36] [2001] for example, and also Jeanblanc & Rutkowski [2002]), such a key point is left for future research along with fault tree and reliability analysis (i.e., general setting for credit risk valuation).

Second, we could extend our sample period in order to incorporate at least two different business cycles (i.e., an economic growth followed by an economic recession or the reverse situation). In this way, we would obtain more realistic estimates since calculated on the two possible states of the world, or equivalently, on two distinct economic scenarii. And, some of the rejected statistical representations (i.e., lognormal, gamma, weibull and beta of second species laws) would certainly become valid in a non-stable economic setting. Indeed, the occurrence of extreme unfavorable events during downturns would increase the bad side of default risk (i.e., fatter left tails due to increased shocks to firms' financial health). Over a longer sample time period (i.e., several business cycles), we could also test the probabilistic representation named fatigue life of Birnbaum & Saunders (1969a,b) to assess French firms' reliability provided that we state the appropriate assumptions and framework. Indeed, this representation assumes repeated cycles of stress scenarii leading to firms' bankruptcy. In such a case, default is no more an absorbing state since any firm can recover from failure and go back to bankruptcy in a 'cyclical' (i.e., iterated) manner. This setting is plausible and realistic as long as default does not imply a liquidation of the firm's assets. Moreover, this framework assumes independency between the current stress cycle and past stress cycles. Such a probabilistic representation allows to characterize whether highly skewed and long tailed lifetimes or nearly symmetric and short tailed lifetimes. Future research should start some reflection about such insights in a more efficient, fine and dynamic risk management prospect. Indeed, default risk assessment's goal is to get in phase with economic, financial and accounting situations.

Finally, we applied our optimal characterizations of French bankruptcies to compute credit spreads in the lens of the reduced form approach of credit risk. Theoretical credit spreads are decreasing functions of both time to maturity and recovery rates (when the later are assumed constant). Incidentally, we underline and establish the clear link prevailing between 'classic' credit risk analysis and the alternative approach of fault tree theory. The next step for future research is to encompass the reduced form side of credit risk analysis in the more general framework of reliability. Hence, credit risk assessment will focus on any chain of events leading to any firm's bankruptcy in a dynamic setting. Such an assessment could be achieved while using some of the well-known technical and stochastic methods peculiar to reliability, namely Petri networks or Markov

[34] Those authors employ Cox-type stochastic processes in a Markov modeling framework (i.e., inhomogeneous Markov chain).

[35] This author applies a doubly stochastic Poisson process to assess credit risky assets (e.g., bonds and credit derivatives) with a fractional recovery rate.

[36] Those authors extend reduced form models to account for default intensities that depend on firm-specific risks, which are considered as counterparty risks.

graphs' theory. Employing reliability to assess credit risk will consequently allow us to finally match both financial, accounting and macroeconomic data. Helping therefore to reconcile structural approach with reduced form approach of credit risk valuation in a more general, flexible, sound and reliable dynamic risk management framework.

7 Appendix

We give some complementary information or details relative to our default risk analysis in this section. First, we give some in sample conditional default probabilities' estimates. Second, we show graphically the possible employment of our framework to assess credit risk.

7.1 Conditional default probabilities

We compute the forward conditional default probabilities on the forthcoming one, two and five year(s) horizons when starting from the beginning of our sample period. Namely, we are computing in-sample forward conditional default probabilities (i.e., only estimation but no forecast of such probabilities).

Table 18: Forward conditional default probabilities (in percent)

Sector	1 year	2 years	5 years
AU	0.1188	0.1188	0.1188
BC	0.0577	0.0577	0.0577
BE	0.0765	0.0765	0.0765
BI	2.4182	2.4417	2.4420
BP	0.0138	0.0138	0.0138
DA	2.7229	3.5904	3.9902
DN	0.0606	0.0606	0.0606
DS	0.0128	0.0128	0.0128
GA	0.0155	0.0155	0.0155
GN	0.0499	0.0499	0.0499
HR	0.1695	0.2158	0.2813
IA	0.0233	0.0233	0.0233
IM	0.0383	0.0383	0.0383
SE	0.1787	0.2286	0.2998
SP	0.2035	0.2577	0.3333
TT	0.1137	0.1137	0.1137
Total	0.6437	0.6437	0.6437

Notice that these forward conditional default probabilities seem to be constant whatever the forthcoming time horizon.[37] This behavior is due to the fact that

[37] We get the same results while employing a ten digits rule for our default probabilities' decimals.

related failure rates decrease quickly towards zero as functions of time. Accordingly, forward conditional default probabilities of BI, DA, HR, SE and SP sectors are increasing functions of coming time horizon while forward conditional default probabilities of remaining sectors are stable over time. Indeed, forward conditional default probabilities of Total sector suggest a stable general trend over 1, 2 and 5 years horizons starting from January 1990. Moreover, among our sixteen economic sectors, only BI and DA sectors exhibit forward conditional default probabilities that lie far above the level of Total sector's forward conditional default probability. Some of the French economic features during our studied decade allow to justify such high probability levels. Indeed, on an annual basis, the number of failures for the industry sector remains first globally stable (i.e., general stable level over ten years). Second, the global trade sector exhibits a number of failures that is higher than other economic sectors during this decade. Specifically, the food branch exhibits a non-negligible number of resounding failures that is probably due to the related reoganization process undergone. Finally, DS sector exhibits the lowest forward conditional default probabilities whereas DA sector exhibits the highest ones.

7.2 An example of application

We show here graphically some dynamic application of our credit risk assessment framework provided to add the appropriate improvements and assumptions (e.g., Petri networks). Let us introduce some definitions before introducing our diagram. We establish the following notations for ease of exposition:

Table 19: Some definitions

Notation	**Meaning**
FC	Financial crisis state
NFC	Non-Financial crisis state
AFF	Accounting and financial factors
NAFF	Non-Accounting and Non-financial factors

Accounting and financial factors are assumed to summarize any relevant information about structural features of firms among others. The general unified framework that could allow to encompass all the approaches of credit risk valuation existing to date is introduced in Fig. 2 below.

The tree's part corresponding to the economic state represents the first level of our risk analysis while considering business cycle's effects. The second level of our risk analysis, as described by FC and NFC, accounts for systematic risk (i.e., high and low levels of the undiversifiable risk that is common to any financial asset). Finally, the third level of our risk study characterizes a specific risk level along with AFF and NAFF variables (i.e., structural, industry and sector specific features as well as operational risk side for example). Notice

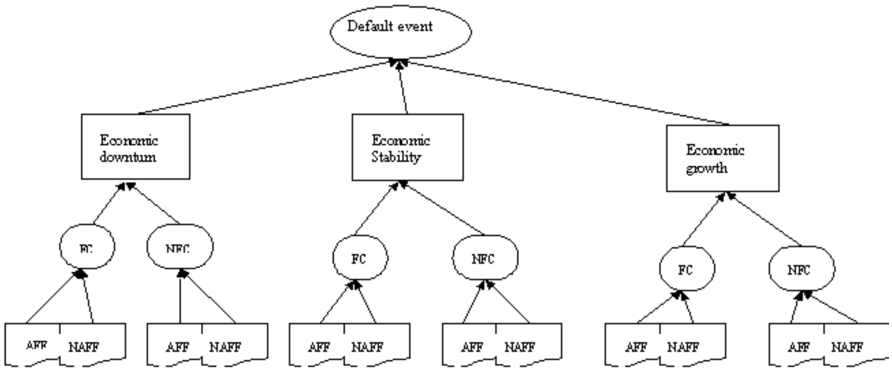

Figure 2: Potential application of dynamic fault tree theory

that a fourth level can be added to account for normative, institutional and legal factors or patterns that describe failure or default state in the accounting, financial or else viewpoints (e.g., failure law, accounting and financial standard). Moreover, the extreme left branch of the second risk level of the tree allows us to characterize two kinds of extreme scenarii (i.e., worst situations for firms). We then have a finest description of the combination of events possibly leading to default. The minimal path leading to bankruptcy allows to incorporate a large number of explanatory variables accounting for both business cycle's effect, systematic risk and specific risk (see Allen & Saunders [2003, 2004] for a brief and clear review about credit risk valuation in the lens of these three dimensions). Both typology and tradeoff between events entering the composition of such a tree describe the credit quality's (i.e., creditworthiness) potential probability of transition from one state to another at a given time and for any specified firm.

Aknowledgements

We are grateful to the participants at the 17^{th} IAE National Days (Lyon, France, September 2004) for their interesting remarks. The usual disclaimer applies.

References

Altman E. I., 1993, *Corporate Financial Distress and Bankruptcy*, 2nd Edition, Wiley Frontiers in Finance.

Allen L. and A. Saunders, 2003, *A Survey of Cyclical Effects in Credit Risk Measurement Model*, BIS Working Papers No. 126.

Allen L. and A. Saunders, 2004, *Incorporating Systematic Influences into Risk Measurement: A Survey of the Literature*, forthcoming in the Journal of Financial Services Research.

Bakshi G., Madan D. & F. Zhang, 2004, *Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates*, Working Paper, 64th AFA Conference, San Diego (USA).

Basel Committee on Banking Supervision, 1996, *Amendments to the Capital Accord to Incorporate Market Risks*, Basel, January.

Birnbaum Z. W. & S. C. Saunders, 1969a, *A New Family of Life Distribution*, Journal of Applied Probability, vol. 6, p. 319 - 327.

Birnbaum Z. W. & S. C. Saunders, 1969b, *Estimation for a Family of Life Distributions with Applications to Fatigue*, Journal of Applied Probability, vol. 6, p. 328 - 347.

Bon J. L., 1995, *Fiabilité des Systèmes*, Masson, Paris.

Cabarbaye A., 1998, *Modélisation et Evaluation des Systèmes*, Cours de Technologies Spatiales, Edition Capadues, Toulouse.

Fisher L., 1959, *Determinants of Risk Premiums on Corporate Bonds*, Journal of Political Economy, vol. 67, n° 3, p. 217 - 237.

Fons J. S. & A. Kimball, 1991, *Corporate Bond Defaults and Default Rates 1970-1990*, Journal of Fixed Income, vol. 1, n° 1, p. 36 - 47.

Fons J. S., 1994, *Using Default Rates to Model the Term Structure of Credit Risk*, Financial Analyst Journal, vol. 50, n° 5, p. 25 - 32.

Gatfaoui H., 2003, *From Fault Tree to Credit Risk Assessment: An Empirical Attempt*, Working Paper, 20th International Conference of the French Finance Association (AFFI), Lyon (France).

Giesecke K. & L. R. Goldberg, 2003, *The Market Price of Credit Risk*, Working Paper, Cornell University.

Gill R. & S. Johansen, 1990, *A Survey of Product-Integration with a View towards Applications in Survival Analysis*, The Annals of Statistics, vol. 18, n° 4, p. 1501 - 1555.

Hamilton J. D., 1994, *Time Series Analysis*, Princeton University Press, NJ.

Hansen L. P., 1982, *Large Sample Properties of Generalized Method of Moments Estimators*, Econometrica, vol. 50, n° 5, p. 1029 - 1054.

Henley E. J. & H. Kumamoto, 1992, *Probabilistic Risk Assessment*, IEEE Press.

Jarrow R. A. & F. Yu, 2001, *Counterparty Risk and the Pricing of Defaultable Securities*, Journal of Finance, vol. 56, n° 5, p. 1765 - 1799.

Jeanblanc M. & M. Rutkowski, 2002, *Default Risk and Hazard Processes*, in Mathematical Finance - Bachelier Congress 2000, Springer-Verlag, Berlin, p. 281 - 312.

Lando D., 1998, *On Cox Processes and Credit Risky Securities*, Review of Derivatives Research, vol. 2, p. 99 - 120.

Mittelhammer R. C., Judge G. G. & D. J. Miller, 2000, *Econometric Foundations*, Cambridge University Press, USA.

Ngom L., Geffroy J. C., Baron C. & A. Cabarbaye, 1999, *Prise en Compte des Relations de Dépendances dans la Simulation de Monte-Carlo des Arbres de Défaillances Non Cohérents*, Phoebus, n° 9 (First quarter), p. 24 - 33.

Papoulis A., 1984, *Probability, Random Variables, and Stochastic Processes*, 2nd Edition, McGraw-Hill, New York.

Polak E., 1971, *Computational Methods in Optimization*, Academic Press, New York.

Press W. H., Teukolsky S. A., Vetterling W. T. & B. P. Flannery, 1992, *Numerical Recipes in C*, 2nd Edition, Cambridge University Press, USA.

Reory K. A. & J. D. Andrews, 2003, *A Fault Tree Analysis Strategy Using Binary Decision Diagrams*, Loughborough University Working Paper (Leicestershire).

Rothenthal A., 1998, *Decomposition Method for Fault Tree Analysis*, IEEE Transactions on Reliability, vol. R-29 (2), p. 136 - 138.

Ruud P. A., 2000, *An Introduction to Classical Econometric Theory*, Oxford University Press, USA.

Tassi P., 1992, *Méthodes Statistiques*, Economica, Paris.