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Reconciling the Nash and Kalai-Smorodinsky Cooperative Solutions: Generalized Maximands of CES Form [†]

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ABSTRACT

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Ana Paula Martins

This note suggests variations to the baseline Nash cooperative solution that take into account the Kalai -Smorodinsky critique. One the one hand, a CES form of the maximand is proven to accommodate both the generalized two -person Nash and the Kalai -Smorodinsky - as other proportional - solutions as special cases.

As an alternative, a Stone -Geary formulation is forwarded, weighing both the distances to the threat and to the ideal point, along with the corresponding CES generalization.

Interpretations of the implied equilibrium solutions – generalizable to n-person cooperative games – are provided, arising as equations balancing geometric averages of measures of attitude towards (large) risk(s) of the players.

JEL Classification: C71; H56; D74 (J51; J52; D39).

Keywords : Two -Person Cooperative Games; Cooperative Games Maximands; Opportunism; Pessimism.

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Introduction

Both the Nash and Kalai -Smorodinsky two -person cooperative games solutions weigh the break-up point of negotiations, specifically, the players' utility in the *status quo*. Yet, only the latter considers explicitly the maximum utility each intervening party can aspire given the possibility set at the bargaining table. It is the purposes of this research to suggest modifications to the standard format of the Nash solution in order to achieve reference to the potential maximal benefits each individual can hope to enjoy with the transaction – the "ideal" utilities of the players.

This quality of an equilibrium solution is philosophically meaningful for two reasons: on the one hand, nature may, in the possibility set, benefit more one than the other player. If that is accounted for in what concerns the absence of transaction in both cited solutions, it is not in terms of potential claims. On the other hand, a total win – a complete victory - may be a reference for each player as important and conditioning as a total loss – and to some extent, this justifies the Kalai-Smorodinsky proposal. However, being their solution a "proportional" one [1] - in the sense that it can be made correspond to the optimization of a (and special) Leontief or fixed coefficients maximand -, it severely conditions the features of the equilibrium solution.

Other maximands have been forwarded in the literature, namely the generalized Gini solution of Blackorby, Bossert and Donaldson (1994). The disadvantage of such (and other) form is lack of mathematical tractability in terms of potential empirical and theoretical applications, namely in labor market analysis. In fact, that rendered the Stone -Geary/Cobb-Douglas structure of the Nash maximand quite popular in the field [2].

A major justification for the use of the Nash maximand came from sequential bargaining rationales: the instantaneous parallel to the Rubinstein (1982) outcome derived by Binmore, Rubinstein and Wolinsky (1986). Others rely on the axiomatic interpretation of the implied equilibrium conditions, namely Svejnar's (1986), relating the ratio between the measures of fear of disagreement of the players to the corresponding bargaining strengths . In this article, we approach this second line of thought.

At a first attempt, a re -definition of the threat point in the standard Nash maximand can be forwarded, namely, with reference to the possible - probable success of - unilateral

[1] See Kalai (1977).

[2] See, for example, Cahuc and Zylberberg (2001) textbook for a recent overview.

appropriation of the bargained object. Weighing the bargaining likelihood against theft/war observation is then obvious.

A second attempt to incorporate ideal utilities is made at the bargaining strength parameter level in the conventional maximand. Another relies on the properties of the CES functional form, which is known to represent both the Cobb-Douglas as the Leontief or fixed coefficient technologies as special cases; with an appropriate parametrization, it is also proven as able to accommodate both Nash and Kalai-Smorodinsky solutions.

Finally, Cobb-Douglas forms factoring the ratios between the distance to the threat point and to the ideal utilities of the several players are inspected as possessing the same potential qualities as the two cooperative paradigms.

Technically, our methodology usually departs from the inspection of the properties of the solutions under an hypothetical egalitarian possibility set, i.e., one offering complete and one-to-one substitution between utilities of the two players, and, whenever possible, highlighting in them the two theoretical paradigms as special cases – Nash and Kalai-Smorodinsky's. Through manipulation of the restricted maximum first-order conditions, relations between indicators akin to risk-aversion measures, drawn from uncertainty theory were also sought, representing qualities possessed by the implied game equilibria.

The exposition proceeds as follows: in section 1, we advance general notation presenting brief sketches of Nash and Kalai-Smorodinsky results. Section 2, derives some implications of the re-definition of the threat-point or status quo in the standard Nash maximand. Section 3 explores the effects of the re-alignment of the bargaining strength parameters. Section 4 forwards the appropriate CES generalization encompassing both the Nash and K-S solutions. Finally, a Cobb-Douglas functional form, and corresponding CES generalization, on arguments weighing both the distance between the utility of each player to his threat and to his ideal point is advanced in section 5. The exposition ends with some concluding remarks.

1. Notation: The Nash (N) and Kalai-Smorodinsky (K-S) Maximands

. Suppose a two-person game, representing negotiations between two parties. Possible pairs of utility [3] for the two individuals, i = 1,2, (u_1, u_2), are restricted to belong to a possibility set G; the *status quo* is an utility pair d = (d_1, d_2) available to the parties in case of negotiation break-down. Nash (1950, 1953) advances four axioms [4] that an optimal allocation - $u^N = (u_1^N, u_2^N) \in G$, denote it by $u^N = f(G, d)$ - solution should exhibit:

[3] We will always assume von Neumann-Morgenstern utilities. Cardinal utility comparability is also required in axiomatic bargaining frameworks, as in most cooperative games.

[4] We follow Cahuc and Zylberberg (2001) exposition of the Nash axioms.

Axiom 1: Pareto Optimality. If $u \in G$ and $u \geq u^N$, then $u = u^N$.

Axiom 2: Invariance with Respect to Positive Linear Transformations of Utility. Let $u^{\#}_i = a_i u_i + b_i$ in which a_i and b_i are constants; if $u^N = (u_1^N, u_2^N) = f(G, d)$, for players having utility functions $u^{\#}_i$, $u^{\#N} = (u^{\#}{}_1^N, u^{\#}{}_2^N) = (a_1 u_1^N + b_1, a_2 u_2^N + b_2)$ is a Nash solution as well.

Axiom 3: Symmetry. If $d_1 = d_2$, and $(u_1, u_2) \in G \Rightarrow (u_2, u_1) \in G$, then $u_1^N = u_2^N$.

Axiom 4: Independence of Irrelevant Alternatives. If $B \subset G$ and $f(G, d) \in B$, then $f(B, d) = f(G, d)$.

Then an optimal solution should satisfy:

$$u^N = \underset{u \in G}{\text{Arg Max}} \ (u_1 - d_1)\ (u_2 - d_2)$$

. Let $\bar{u}_1$ and $\bar{u}_2$ denote the maximum utilities that each agent can achieve, regardless of the other's position, with the transaction, i.e., within set G – their "ideal" utilities in the possibility set G; denote u^K the bargaining solution for set G, and $u^{K'}$ the solution under set G' - with maximums $\bar{u}_1$' e $\bar{u}_2$'. Then, Kalai and Smorodinsky (1975), replacing Axiom 4 by:

Axiom 5: Individual Monotonicity [5]. If $G \subset G'$ and $\bar{u}_1 = \bar{u}_1'$, then $u_2^{K'} > u_2^K$.

conclude that the optimal solution u^K should be the point of set G more to the northeast - in the (u_1, u_2) space - satisfying the proportionality [6] condition:

$$\text{(1)} \qquad \frac{u_2^K - d_2}{u_1^K - d_1} = \frac{\bar{u}_2 - d_2}{\bar{u}_1 - d_1}$$

That is, their maximand can be seen to be represented by a Leontief objective function:

$$\text{(2)} \qquad u^K = \underset{u \in G}{\text{Arg Max}} \ \text{Min}\left(\frac{u_1 - d_1}{\bar{u}_1 - d_1}, \frac{u_2 - d_2}{\bar{u}_2 - d_2}\right)$$

that is, it satisfies a max-min criteria for game resolution [7].

. Nash and Kalai-Smorodinsky solutions can be visualized in Figure 1 below. Let the points in G, the area bordered by the bold line, represent the available bargaining set; (d_1, d_2) denote the status quo utilities - they could as well be contained in set G. $\bar{u}_1$ is the coordinate in

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[5] See Riddell (1981).

[6] See also Kalai (1977).

[7] See Luce and Raiffa (1957) and Roth (1977); also Kalai (1977).

the horizontal axis of the most eastern point in G; $\overline{u}_2$ is the ordinate of the most northern point in G.

- Consider rectangular hyperbolas in the axis formed by (d_1, d_2) – the pseudo social indifference curves of generic form $(u_1 - d_1)(u_2 - d_2) = \overline{\varpi}$; the Nash solution is the point, N, in G touched by the most northeastern of those hyperbolas.

- Kalai-Somorodinsky outcome, K, is the most northeastern point in G on the straight line connecting (d_1, d_2) to $(\overline{u}_1, \overline{u}_2)$. Hence, in the most northeastern "social indifference curve" of the form $\mathrm{Min}\left(\frac{u_1 - d_1}{\overline{u}_1 - d_1}, \frac{u_2 - d_2}{\overline{u}_2 - d_2}\right) = \overline{\varpi}$ touching G.

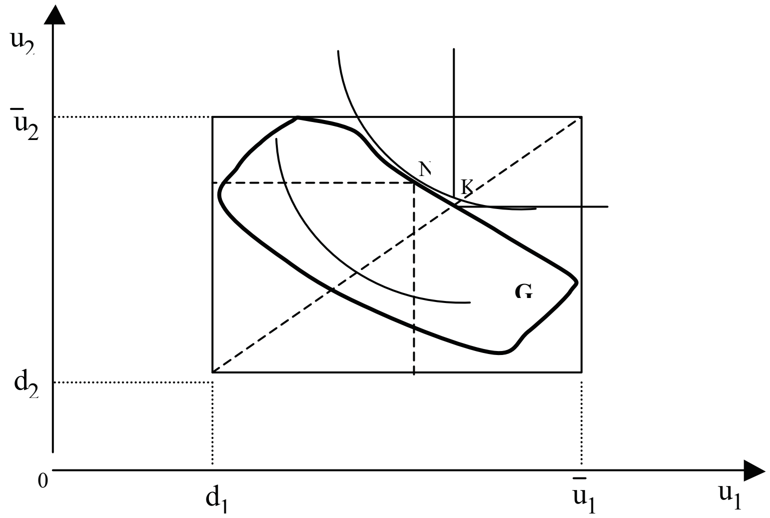

Figure 1. Nash and Kalai-Smorodinsky's Solutions [8]

. Generalizations of the Nash solution admit some asymmetry, proposing:

$$(3) \qquad u^{NG} = \underset{u \in G}{\mathrm{Arg\,Max}}\ (u_1 - d_1)^{\gamma 1}\ (u_2 - d_2)^{\gamma 2}$$

where $\frac{\gamma_i}{\gamma_1 + \gamma_2}$ represents bargaining power of player i. Sequential bargaining equivalences approximate γ_i as proportional to the inverse of player's i interest (discount) rate.

Under a production function – as under a standard consumer utility function – the substitutability between any two inputs is associated to the marginal rate of substitution between them. A measure of interpersonal comparison of utilities under a social welfare function could be associated to the implicit marginal rate of substitution between them, to the curvature of such isoquants. Consider the MRS between u_1 and u_2 in the above maximand, $\Omega(u_1, u_2)$, the absolute value of the slope of an indifference curve at a particular point in space (u_1, u_2):

[8] See Kalai and Smorodinsky (1975). We superimpose the underlying indifference curves.

(4) $$\mathrm{MRS}_{u_1,u_2} = \frac{\frac{\partial\Omega}{\partial u_1}}{\frac{\partial\Omega}{\partial u_2}} = -\frac{du_2}{du_1}\Big|_{\Omega=\varpi} = \frac{\gamma_1}{\gamma_2}\ \frac{u_2-d_2}{u_1-d_1}$$

One can inspect the characteristics of an optimal allocation for an additive possibility set, i.e., one along which border individual utilities face a unitary trade -off. That is, if points in G obey:

(5) $$\sum_{i=1}^{n} u_i \le \overline{U}$$

where $\overline{U}$ is a constant - which implies an equal distribution of utility possibilities –the implied distribution of utilities under a given bargaining maximand is deriv ed from the condition:

(6) $$\mathrm{MRS}_{u_1,u_2} = 1$$

For the above maximand it requires:

(7) $$\frac{u_2-d_2}{u_1-d_1} = \frac{\gamma_2}{\gamma_1}$$

The ratio between utility increments obtained with the transaction by the two players is going to be equal to t heir relative bargaining strength. Conjugating with the egalitarian border equation, we can derive that:

(8) $$u_1^{NG} - d_1 = \frac{\gamma_1}{\gamma_1+\gamma_2}\ (\overline{U}-d_1-d_2) \quad \text{or} \quad u_1^{NG} = \frac{\gamma_1(\overline{U}-d_2)+\gamma_2 d_1}{\gamma_1+\gamma_2}$$

The increment of utility ach ieved by individual i is going to be a share of the total incremental utility under distribution to both players equal to his bargaining strength.

Let us consider now the K -S the optimal solution for the egalitarian possibility set. Under $u_1 + u_2 \le \overline{U}$:

(9) $$u_1^{K} - d_1 = \frac{\overline{u}_1-d_1}{\overline{u}_1-d_1+\overline{u}_2-d_2}\ (\overline{U}-d_1-d_2) \quad \text{or} \quad u_1^{K} = \frac{(\overline{u}_1-d_1)(\overline{U}-d_2)+(\overline{u}_2-d_2)d_1}{(\overline{u}_1-d_1)+(\overline{u}_2-d_2)}$$

The share obtained by player i has correspondence with the weight of the maximal utility he can hope for with the sum of such utility increment potential for all players.

Under the assumption of individual monotonicity, $\overline{u}_i$ is the maximal utility for player i in set G, regardless of the other player's utility; under $u_1 + u_2 \le \overline{U}$, and a requirement of strictly non-negative utilities, that implies $\overline{u}_i = \overline{U}$. Then:

(10) $u_1^K - d_1 = \dfrac{\overline{U} - d_1}{\overline{U} - d_1 + \overline{U} - d_2}\,(\overline{U} - d_1 - d_2)$ or $u_1^K = \dfrac{\overline{U} - d_2}{(\overline{U} - d_1) + (\overline{U} - d_2)}\,\overline{U}$

In some situations, however, it could be the case that only points to the northeast of (d_1, d_2) are in the current set G. Then, $\overline{u}_i = \overline{U} - d_j$.

(11) $u_1^K - d_1 = \dfrac{\overline{U} - d_1 - d_2}{2}$ or $u_1^K = \dfrac{\overline{U} + d_1 - d_2}{2}$

This was not necessarily the spirit of the K-S monotonicity assumption, under which neither (d_1, $\overline{u}_2$) nor ($\overline{u}_1$, d_2) – nor even (d_1, d_2) - had to belong to set G; it is understood in most of the literature that G is defined in the range to the northeast of (d_1, d_2) [9], but that is not a mathematical requirement for the properties of the equilibrium – or of any proportional solution [10] - to hold. Depending on the bargaining conditions – on the bargained set G and perceived ideal points– one or the other case may be applicable.

. We encounter another justification of the generalized Nash maximand in Svejnar (1986), who considers an extension to a n-person game. Admit that what is at stake is the allocation x_i, i=1,2,…,n, among the players of a total amount X = $\sum_{i=1}^{n} x_i$, and that individuals' utility functions are (differentiable, etc.) represented by $u_i(x_i)$. He defines *fear of disagreement* of player i at a given allocation is defined as:

(12) $$f_i(x_i) = \frac{u_i(x_i) - d_i}{u_i'(x_i)}$$

measuring how an already attained incremental utility is exchanged per marginal utility, i.e., per incremental utility per unit of x_i. It assesses an individual's aversion to large risks [11].

Svejnar requires that an axiomatic solution equates the ratios between fear of disagreement of any two players to the ratio of the corresponding – exogenously given – bargaining powers:

[9] See Riddell (1981); Osborne and Rubinstein (1990), p. 22.

[10] See Kalai-Smorodinsky (1977).

[11] In correspondence to *fear of ruin* of Aumann and Kurz (1977), the inverse of *boldness*, the semi-elasticity of the utility function with respect to the argument.

$$(13) \qquad \frac{f_i}{f_j} = \frac{\dfrac{u_i(x_i)-d_i}{u_i'(x_i)}}{\dfrac{u_j(x_j)-d_j}{u_j'(x_j)}} = \frac{\gamma_i}{\gamma_j} \qquad \text{or} \qquad \frac{f_i}{\gamma_i} = \frac{f_j}{\gamma_j}$$

Hence, the generalized Nash maximand is justified.

2. Unilateral Appropriation: Bargaining versus War

. A straight - forward application of Nash prin ciple relating the bargaining objective function to the maximal benefits could be rationalized easily assuming a "threat" point that would weight, at a subjective probability p_i, the possibility of, say, appropriation of the total "cake" by player i. In ge neral, one would expect $0 \le p_i \le 1$ and, for two players, $p_1 + p_2 \le 1$ – together, the two players cannot possibly expect to appropriate unilaterally more than the full cake.

Then, the maximand would become:

$$(14) \quad u^W = \underset{u \in G}{\text{Arg Max}} \ \{u_1 - [p_1 \bar{u}_1 + (1-p_1)\, d_1]\}^{\gamma_1} \{u_2 - [p_2 \bar{u}_2 + (1-p_2)\, d_2]\}^{\gamma_2}$$

Consider the MRS between u_1 and u_2 in the above maximand:

$$MRS_{u_1,u_2} = \frac{\dfrac{\partial \Omega}{\partial u_1}}{\dfrac{\partial \Omega}{\partial u_2}} = \frac{\gamma_1}{\gamma_2} \frac{u_2 - [p_2 \bar{u}_2 + (1-p_2) d_2]}{u_1 - [p_1 \bar{u}_1 + (1-p_1) d_1]}$$

It is straight-forward to derive that: $\dfrac{\partial MRS_{u_1,u_2}}{\partial \gamma_1} > 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial d_1} > 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_1} > 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial p_1} > 0$. In space (u_1, u_2), at a given point the slope of an indifference curve that crosses that point increases with the bargaining power of agent 1, his status quo utility, its "dream utility" $\bar{u}_1$ and the probability of attaining it independently. That implies that if any of those increases is actually observed, the slope of the indifference curve at a previous optimum point in G rises; this suggests - under convex "social indifference curves" - that the new optimal allocation should move to the southeast of the original one – favoring 1 at the expense of 2 relative to the old solution.

Consistently, $\dfrac{\partial MRS_{u_1,u_2}}{\partial \gamma_2} < 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial d_2} < 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_2} < 0$; $\dfrac{\partial MRS_{u_1,u_2}}{\partial p_2} < 0$ and 2's position is expected to improve with either increases in his own opportunities.

Manipulating the condition $MRS_{u_1,u_2} = 1$, we arrive at:

$$(15) \qquad \frac{u_2 - [p_2\bar{u}_2 + (1-p_2)d_2]}{u_1 - [p_1\bar{u}_1 + (1-p_1)d_1]} = \frac{u_2 - [d_2 + p_2(\bar{u}_2 - d_2)]}{u_1 - [d_1 + p_1(\bar{u}_1 - d_1]} = \frac{\gamma_2}{\gamma_1}$$

Under the egalitarian opportunity set, $u_1 + u_2 \leq \bar{U}$:

$$(16) \quad u_1^W - [p_1 \bar{u}_1 + (1-p_1) d_1] = \frac{\gamma_1}{\gamma_1+\gamma_2} \{\bar{U} - d_1 - d_2 - [p_1(\bar{u}_1 - d_1) + p_2(\bar{u}_2 - d_2)]\}$$

$$u_1^W - d_1 = \frac{\gamma_1(\bar{U} - d_1 - d_2) - \gamma_1 p_2(\bar{u}_2 - d_2) + \gamma_2 p_1(\bar{u}_1 - d_1)}{\gamma_1 + \gamma_2}$$

$$u_1^W = \frac{\gamma_1\{\bar{U} - [d_2 + p_2(\bar{u}_2 - d_2)]\} + \gamma_2[d_1 + p_1(\bar{u}_1 - d_1)]}{\gamma_1 + \gamma_2}$$

One can show that:

$$(17) \qquad u_1^W > u_1^{NG} \text{ of } (8) \qquad \text{iff } \frac{p_1(\bar{u}_1 - d_1)}{p_2(\bar{u}_2 - d_2)} > \frac{\gamma_1}{\gamma_2}$$

that is, i will now be better -off if and only if the ratio between the product of the probability of successful unilateral appropriation and the maximal surplus of both players is larger that their relative bargaining strength.

If we admit that $\bar{u}_1 = \bar{u}_2 = \bar{U}$, one can derive:

$$(17) \qquad u_1^W - [p_1 \bar{U} + (1-p_1) d_1] = \gamma_1 \frac{\bar{U}(1-p_1-p_2) - (1-p_1)d_1 - (1-p_2)d_2}{\gamma_1 + \gamma_2}$$

$$u_1^W - d_1 = \frac{\gamma_1[(1-p_2)(\bar{U} - d_2) - d_1] + \gamma_2 p_1(\bar{U} - d_1)}{\gamma_1 + \gamma_2}$$

$$u_1^W = \frac{\gamma_1(1-p_2)(\bar{U} - d_2) + \gamma_2[p_1\bar{U} + (1-p_1)d_1]}{\gamma_1 + \gamma_2}$$

. The solution has the merit of specifically contemplating both the minimum as maximum possible states, with each trader's position in the final outcome being affected by both in the same direction. Yet, two disadvantages of the approach are obvious: one, for sufficiently high p_i's, it may be impossible to attain a solution where $\{u_i - [p_i \bar{u}_i + (1-p_i) d_i]\}$ are both larger than 0 – or, equivalently, $\{[p_1 \bar{u}_1 + (1-p_1) d_1], [p_2 \bar{u}_2 + (1-p_2) d_2]\}$ may not belong (that is, rather be to the northeast) to the bargaining possibility set G; if it does not, both players will "agree" on engaging in "war". Two, it raises the philosophical question of whether a "status quo" (d_1, d_2) is a meaningful threat – if in certain games it is not, then the current maximand would be more appropriate.

Admit again the egalitarian opportunity set and that $\bar{u}_1 = \bar{u}_2 = \bar{U}$. For $\{[p_1 \bar{u}_1 + (1 - p_1) d_1], [p_2 \bar{u}_2 + (1 - p_2) d_2]\}$ to belong to G:

(18) $$\bar{U}(1 - p_1 - p_2) - (1 - p_1)d_1 - (1 - p_2)d_2 \geq 0$$

From the perspective of player 1, that requires t hat:

(19) $$p_1 \leq \frac{(\bar{U} - d_2)(1 - p_2) - d_1}{\bar{U} - d_1} = 1 - \frac{p_2\bar{U} + (1 - p_2)d_2}{\bar{U} - d_1} = p^L_1$$

The higher this limiting probability of successful appropriation by individual 1, p^L_1, the more likely will bargaining be observed and not "war". *Coeteris paribus*, p^L_1:

- decreases with the probability with which the opponent perceives success in a "war", in obtaining the full "pie", p_2.
- decreases with both his and the opponent's *status quo* endowments, d_1 and d_2.
- increases with the possibility set range, that is wi th $\bar{U}$.

Symmetric inference could be drawn for 2. We can thus conclude that:

Proposition 1: Negotiations will more likely break -down:

1.1. The higher the subjective probability of a total unilateral victory perceived by each party

1.2. The higher the *status quo* endowment of each party

1.3. The lower the available surplus (the total endowment) at the bargaining table.

In some cases – and making analogy with the discussion of "ideal" utility of each player in the K-S solution – it is possible that $\bar{u}_i = \bar{U} - d_j$. Then $MRS_{u_1,u_2} = 1$, (15), originates:

$$\frac{u_2 - [p_2(\bar{U} - d_1 - d_2) + d_2]}{u_1 - [p_1(\bar{U} - d_1 - d_2) + d_1]} = \frac{\gamma_2}{\gamma_1}$$

and

(20) $$u_1^W - [p_1(\bar{U} - d_1 - d_2) + d_1] = \gamma_1 \frac{(1 - p_1 - p_2)(\bar{U} - d_1 - d_2)}{\gamma_1 + \gamma_2}$$

$$u_1^W - d_1 = \frac{[\gamma_1(1 - p_2) + \gamma_2 p_1](\bar{U} - d_1 - d_2)}{\gamma_1 + \gamma_2}$$

$$u_1^W = \frac{[\gamma_1(1 - p_2) + \gamma_2 p_1](\bar{U} - d_2) + [\gamma_1 p_2 + \gamma_2(1 - p_1)]d_1}{\gamma_1 + \gamma_2}$$

Now, for [$p_1(\overline{U} - d_1 - d_2) + d_1,\ p_2(\overline{U} - d_1 - d_2) + d_2$] to belong to set G it suffices that $p_1 + p_2 \leq 1$: the two players together cannot expect to be able to appropriate unilaterally more than the total existing endowment $\overline{U}$.

3. Bargaining Strength Adjustments

. An alternative formulation could consider factoring each individual bargaining strength in the generalized Nash maximand by the maximal increment utility he can hope to achieve, as is with the inverse to the individuals' discount rates, i.e., assume the optimal split obeys:

$$u^W = \underset{u \in G}{\text{Arg Max}} \ (u_1 - d_1)^{\frac{(\overline{u}_1 - d_1)}{r_1}} (u_2 - d_2)^{\frac{(\overline{u}_2 - d_2)}{r_2}} \tag{21}$$

$MRS_{u_1,u_2} = 1$ implies:

$$\frac{u_2 - d_2}{u_1 - d_1} = \frac{\overline{u}_2 - d_2}{\overline{u}_1 - d_1} \frac{r_1}{r_2} = \frac{\gamma_2}{\gamma_1}$$

and we could argue that only a re -scaling of the traditional relative bargaining strength parameter was put in place.

. In correspondence to Svejnar's specification, $\gamma_i = \dfrac{\overline{u}_i - d_i}{r_i}$. i's relative bargaining strength is $\dfrac{\frac{\overline{u}_i - d_i}{r_i}}{\frac{\overline{u}_1 - d_1}{r_1} + \frac{\overline{u}_2 - d_2}{r_2}}$: it increases with $\overline{u}_i$; it decreases with r_i; however, it also decreases with d_i – which nevertheless may be offset by its additional inclusion in the maximand. Considering a splitting X = $x_1 + x_2$ scenario, and that individuals' utility functions are (differentiable, etc.) represented by $u_i(x_i)$, F.O.C. would lead to:

$$\frac{f_i}{f_j} = \frac{\frac{u_i(x_i) - d_i}{u_i'(x_i)}}{\frac{u_j(x_j) - d_j}{u_j'(x_j)}} = \frac{\gamma_i}{\gamma_j} = \frac{\overline{u}_i - d_i}{\overline{u}_j - d_j} \frac{r_j}{r_i} \tag{22}$$

For linear utility functions, and equal discount rates, it is clear that the solution reverts to that of the K -S maximand. For symmetric potential incremental utilities, we arrive at the Nash solution; yet, this would purely be circumstantial, and we had rather accommodate both cases in a more general way:

Proposition 2: A maximand of form (21) can be rationalized by requiring axiomatically that, in the optimal solution, the ratio between fear of disagreement between any two players equates the product of the ratio between the maximal potential increments by the inverse of the ratio of the individuals' interest r ates.

4. A CES Generalization of the Bargaining Maximand

. A third approach would advance a functional form which specifically entails both the Cobb-Douglas as the Leontief formats as special cases, the Constant Elasticity of Substitution being a natura l choice. As is well -known, under two inputs, or CES (or isoelastic, its transformation) social welfare functions, the elasticity of substitution equals $\dfrac{\partial \ln(\frac{u_2}{u_1})}{\partial \ln MRS_{u_1,u_2}}$ at any particular point of a social indifference curve and stands as a measu re of societies taste for inequality of utilities – see Boadway and Bruce (1984), p. 141 -142 and 160. With an appropriate scaling of coefficients, one can recover in this form both N and K -S solutions for the special cases of the elasticity, 0 and 1, corre sponding, respectively to the Leontief and Cobb -Douglas functional forms:

$$(23) \qquad u^W = \underset{u \in G}{\text{Arg Max}} \left\{ a_1 \, [c_1 (u_1 - d_1)]^{\frac{\sigma-1}{\sigma}} + a_2 \, [c_2 (u_2 - d_2)]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

where $a_i = \dfrac{\frac{1}{r_i}}{\frac{1}{r_1}+\frac{1}{r_2}} > 0$, $c_i = \dfrac{1}{u_i - d_i}$, i = 1,2, and $0 \le \sigma \le 1$.

Following the previous inspection, the MRS between the two player's utility can be written as:

$$(24) \quad MRS_{u_1,u_2} = \frac{\frac{\partial \Omega}{\partial u_1}}{\frac{\partial \Omega}{\partial u_2}} = \frac{a_1}{a_2} \left(\frac{c_1}{c_2}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{u_1 - d_1}{u_2 - d_2}\right)^{-\frac{1}{\sigma}} = \frac{a_1}{a_2} \, \frac{c_1}{c_2} \left[\frac{c_1 \, (u_1 - d_1)}{c_2 \, (u_2 - d_2)}\right]^{-\frac{1}{\sigma}}$$

For $0 \leq \sigma \leq 1$, $\frac{\partial MRS_{u_1,u_2}}{\partial a_1} > 0$; $\frac{\partial MRS_{u_1,u_2}}{\partial d_1} > 0$; $\frac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_1} > 0$ and $\frac{\partial MRS_{u_1,u_2}}{\partial a_2} < 0$; $\frac{\partial MRS_{u_1,u_2}}{\partial d_2} < 0$; $\frac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_2} < 0$.

$MRS_{u_1,u_2} = 1$ implies the distribution pattern:

$$(25) \qquad \frac{u_1 - d_1}{u_2 - d_2} = \left(\frac{a_1}{a_2}\right)^{\sigma} \left(\frac{c_1}{c_2}\right)^{(\sigma-1)} = \left(\frac{a_1}{a_2}\right)^{\sigma} \left(\frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}\right)^{(1-\sigma)}$$

The solution ratio between incremental utilities is a weighted geometric mean of $\frac{a_1}{a_2} = \frac{r_2}{r_1}$ and the ratio of maximum incremental utilities. The relative share accruing to 1 will increase with σ iff $\frac{a_1}{a_2} > \frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}$.

Admit an egalitarian opportunity set and replace $u_1 + u_2 = \bar{U}$; one derives:

$$(26) \qquad u_1 - d_1 = \frac{a_1^{\sigma} (\bar{u}_1 - d_1)^{(1-\sigma)}}{a_1^{\sigma} (\bar{u}_1 - d_1)^{(1-\sigma)} + a_2^{\sigma} (\bar{u}_2 - d_2)^{(1-\sigma)}} (\bar{U} - d_1 - d_2) \quad \text{or}$$

$$u_1 = \frac{a_1^{\sigma} (\bar{u}_1 - d_1)^{(1-\sigma)} (\bar{U} - d_2) + a_2^{\sigma} (\bar{u}_2 - d_2)^{(1-\sigma)} d_1}{a_1^{\sigma} (\bar{u}_1 - d_1)^{(1-\sigma)} + a_2^{\sigma} (\bar{u}_2 - d_2)^{(1-\sigma)}}$$

Under equal "financial" strength, $a_1 = a_2$:

$$(27) \qquad u_1 - d_1 = \frac{(\bar{u}_1 - d_1)^{(1-\sigma)}}{(\bar{u}_1 - d_1)^{(1-\sigma)} + (\bar{u}_2 - d_2)^{(1-\sigma)}} (\bar{U} - d_1 - d_2) \quad \text{or}$$

$$u_1 = \frac{(\bar{u}_1 - d_1)^{(1-\sigma)} (\bar{U} - d_2) + (\bar{u}_2 - d_2)^{(1-\sigma)} d_1}{(\bar{u}_1 - d_1)^{(1-\sigma)} + (\bar{u}_2 - d_2)^{(1-\sigma)}}$$

Then, it is straight-forward to conclude that as σ rises to 1, the share of the surplus relative to the *status quo* obtained by player 1 increases (decreases) - from $\frac{(\bar{u}_1 - d_1)^{(1-\sigma)}}{(\bar{u}_1 - d_1)^{(1-\sigma)} + (\bar{u}_2 - d_2)^{(1-\sigma)}}$ to $\frac{1}{2}$ - provided $\frac{(\bar{u}_1 - d_1)^{(1-\sigma)}}{(\bar{u}_1 - d_1)^{(1-\sigma)} + (\bar{u}_2 - d_2)^{(1-\sigma)}} < (>) \frac{1}{2}$, i.e., $\bar{u}_1 - d_1 <$ (>) $\bar{u}_2 - d_2$ - 1's ideal surplus is lower (higher) than 2's.

That is, if $a_1 = a_2$, a rise in σ originates a more egalitarian split of the surplus relative to the *status quo* at the bargaining table: interestingly, and unlike in conventional social welfare isoelastic forms, a rise in the elasticity of substitution (in the 0 -1 range) is expected to generate a more egalitarian distribution of the utility surplus being bargained – the reason being that in the current form of the maximand the reference point entails an unequal ideal surplus distribution.

If $\overline{u}_1 = \overline{u}_2 = \overline{U}$, we can replace both ideal utilities in the expression above by $\overline{U}$ and the conclusions rest unaltered; additionally, $\overline{u}_1 - d_1 < (>)\ \overline{u}_2 - d_2$ - 1's ideal surplus is lower (higher) than 2's iff $d_1 > (<)\ d_2$. If instead, we postulate that $\overline{u}_i = \overline{U} - d_j$:

$$(28) \qquad u_1 - d_1 = \frac{a_1^\sigma}{a_1^\sigma + a_2^\sigma}\,(\overline{U} - d_1 - d_2) \qquad \text{or} \qquad u_1 = \frac{a_1^\sigma(\overline{U} - d_2) + a_2^\sigma d_1}{a_1^\sigma + a_2^\sigma}$$

As σ decreases to 0, the share of the surplus relative to the *status quo* obtained by player 1 increases (decreases) - from $\frac{a_1^\sigma}{a_1^\sigma + a_2^\sigma}$ to $\frac{1}{2}$ - provided $\frac{a_1^\sigma}{a_1^\sigma + a_2^\sigma} < (>)\ \frac{1}{2}$, i.e., $a_1^\sigma < (>)\ a_2^\sigma$ - 1's financial strength is lower than 2's. That is, under an egalitarian opportunity set, with ideal utilities originating points to be to the right of (d_1, d_2), a rise in σ implies higher inequality in the split of the bargaining gains.

Under equal financial strength of the players, we always have

$$(29) \qquad u_1 - d_1 = \frac{1}{2}\,(\overline{U} - d_1 - d_2) \qquad \text{or} \qquad u_1 = \frac{\overline{U} - d_2 + d_1}{2}$$

. Admit the game will split among the players a total amount $X = \sum_{i=1}^{n} x_i$, and that individuals' utility functions are (differentiable, etc.) represented by $u_i(x_i)$. We can interpret the splitting outcome as a weighted geometric mean of both the K-S and N solution formats, provided we restrict the relevant range of σ to the [0, 1] interval:

Manipulating the F.O.C., we can derive:

$$(30) \quad \frac{u_i(x_i) - d_i}{u_j(x_j) - d_j} = \left(\frac{a_i}{a_j}\right)^{\sigma} \left[\frac{u'_i(x_i)}{u'_j(x_j)}\right]^{\sigma} \left(\frac{c_i}{c_j}\right)^{(\sigma-1)} = \left(\frac{a_i}{a_j}\right)^{\sigma} \left[\frac{u'_i(x_i)}{u'_j(x_j)}\right]^{\sigma} \left(\frac{\overline{u}_i - d_i}{\overline{u}_j - d_j}\right)^{(1-\sigma)}$$

The ratio between the incremental gains in the optimal solution of any two players is a weighted geometric mean of the ratio between the maximum potential utility gains of the two players and the ratio of the marginal utilities at the optimal solution – these weighted by relative bargaining power factors. The closer σ is to 0, the more important the ratio of potential increments will be to determine the solution; the closer it is to 1, the more important the relative financial strength will.

One can re-arrange the previous expression in order to isolate the ratio between fear of disagreement in the left hand-side of the expression:

$$(31)\quad \frac{f_i}{f_j} = \frac{\dfrac{u_i(x_i)-d_i}{u_i'(x_i)}}{\dfrac{u_j(x_j)-d_j}{u_j'(x_j)}} = \left(\frac{a_i}{a_j}\right)^{\sigma}\left[\frac{u'_i(x_i)}{u'_j(x_j)}\right]^{(\sigma-1)}\left(\frac{c_i}{c_j}\right)^{(\sigma-1)} = \left(\frac{a_i}{a_j}\right)^{\sigma}\left[\frac{\dfrac{\bar{u}_i-d_i}{u_i'(x_i)}}{\dfrac{\bar{u}_j-d_j}{u_j'(x_j)}}\right]^{(1-\sigma)}$$

. Define *opportunism* at a given allocation x_i, $o_i(x_i)$, as

$$(32)\qquad o_i(x_i) = \frac{\underset{u\in G}{Max}\,u_i - d_i}{u_i'(x_i)} = \frac{\bar{u}_i - d_i}{u_i'(x_i)}$$

It measures how the maximum attainable incremental utility is exchanged per marginal utility, i.e., per infinitesimal incremental utility per unit of x_i. The larger $o_i(x)$ achieved, in better position an individual is in what concerns the possibility set, and the more is he able/expected to exchange incremental utility relative to the amount measured by marginal utility.

We can re-interpret the previous identity: the ratio between the fear of disagreement of any two players is going to be a weighted geometric mean of their relative bargaining strength, $\frac{a_i}{a_j}$, and the ratio of what we can term the measures of *opportunism* of the individuals, $\frac{o_i(x_i)}{o_j(x_j)}$.

Hence, we can rationalize the above CES functional form concluding that $(1-\sigma)$ weights opportunism in the underlying maximand:

$$(33)\qquad \frac{f_i(x_i)}{f_j(x_j)} = \left(\frac{a_i}{a_j}\right)^{\sigma}\left[\frac{o_i(x_i)}{o_j(x_j)}\right]^{(1-\sigma)}$$

Proposition 3: A CES bargaining maximand of form (23), for $0 \le \sigma \le 1$:

3.1. can be rationalized by requiring that in the optimal solution the ratio between fear of disagreement of any two players equalizes the geometric mean of their relative bargaining strength and the ratio of their measures of opportunism with respect to the bargained set.

3.2. under equal "financial strength" of the players, is expected to generate a more egalitarian distribution of the utility surplus being bargained the larger the elasticity of substitution (in the 0-1 range).

3.3. under different "financial strength" of the players and a threat point conditioning the ideal utilities, is expected to generate a less egalitarian distribution of the utility surplus being bargained the larger the elasticity of substitution (in the 0-1 range).

. Other manipulations of the F.O.C. allow the ratio between the incremental utilities of any two players relative to the *status quo* and the maximum gain with the transaction to be written as:

$$\frac{\dfrac{u_i(x_i)-d_i}{\overline{u}_i-d_i}}{\dfrac{u_j(x_j)-d_j}{\overline{u}_j-d_j}} = \left[\frac{a_i\,\dfrac{u'_i(x_i)}{(\overline{u}_i-d_i)}}{a_j\,\dfrac{u'_j(x_j)}{(\overline{u}_j-d_j)}}\right]^{\sigma}$$

We can re-arrange terms in order to obtain:

$$\frac{a_i\,\dfrac{u'_i(x_i)}{(\overline{u}_i-d_i)}}{a_j\,\dfrac{u'_j(x_j)}{(\overline{u}_j-d_j)}} = \left[\frac{\dfrac{u_i(x_i)-d_i}{\overline{u}_i-d_i}}{\dfrac{u_j(x_j)-d_j}{\overline{u}_j-d_j}}\right]^{\frac{1}{\sigma}}$$

$$\frac{\dfrac{(\overline{u}_i-d_i)}{u'_i(x_i)}}{\dfrac{(\overline{u}_j-d_j)}{u'_j(x_j)}} = \frac{a_i}{a_j}\left[\frac{\dfrac{\overline{u}_i-d_i}{u_i(x_i)-d_i}}{\dfrac{\overline{u}_j-d_j}{u_j(x_j)-d_j}}\right]^{\frac{1}{\sigma}}$$

. One may wish to interpret solutions for which $\sigma > 1$ – *a priori*, the general form (23) would encompass such case and, in general, the subsequent expressions would still be valid.

A first comment we may forward is that for $\sigma > 1$, and opposite to what happens for values of σ in the 0-1 range, $\dfrac{\partial MRS_{u_1,u_2}}{\partial \overline{u}_1} < 0$ and $\dfrac{\partial MRS_{u_1,u_2}}{\partial \overline{u}_2} > 0$. This will condition the general properties of the solution. For instance, by inspection of (26), it is easily concluded that under an egalitarian opportunity set, as σ tends to ∞, the player with higher $\dfrac{a_i}{\overline{u}_i-d_i}$ would tend to obtain (as surplus with respect to the status quo) the whole surplus $(\overline{U}-d_1-d_2)$ - interestingly, high expectations or ideals become harmful for the player. Hence, large values of σ could reproduce highly unbalanced distributions of the bargaining gains.

One can re-arrange (33) to:

(34) $$\left[\frac{f_i(x_i)}{f_j(x_j)}\right]^{\frac{1}{\sigma}}\left[\frac{o_i(x_i)}{o_j(x_j)}\right]^{\frac{\sigma-1}{\sigma}} = \frac{a_i}{a_j}$$

For values of $\sigma > 1$, the F.O.C. would dictate an equality of the weighted geometric mean of relative fear of disagreement – with larger weight given to this the lower σ - and relative opportunism to the relative "financial strength" of the players.

Proposition 4: A CES bargaining maximand of form (23), for $\sigma > 1$:

4.1. can be rationalized by requiring that, in the optimal solution, the weighted geometric mean of the ratio between fear of disagreement of any two players and the ratio of their measures of opportunism with respect to the bargained set equalizes their relative bargaining strength.

4.2. is expected to generate a less egalitarian distribution of the utility surplus being bargained the larger the elasticity of substitution.

With a simple modification to (30), one can recover the previous signs $\frac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_1} > 0$ and $\frac{\partial MRS_{u_1,u_2}}{\partial \bar{u}_2} < 0$ for all values of σ considering:

$$u^{W'} = \underset{u \in G}{\text{Arg Max}} \left\{ a_1 \, [b_1^{\sigma-1}(u_1 - d_1)]^{\frac{\sigma-1}{\sigma}} + a_2 \, [b_2^{\sigma-1}(u_2 - d_2)]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

where $a_i = \frac{\frac{1}{r_i}}{\frac{1}{r_1} + \frac{1}{r_2}} > 0$, $b_i = \bar{u}_i - d_i$, $i = 1,2$, and $\sigma \geq 0$. Under the above function, in an internal solution of the splitting "pie" problem:

$$\frac{f_i}{f_j} = \frac{\frac{u_i(x_i) - d_i}{u_i'(x_i)}}{\frac{u_j(x_j) - d_j}{u_j'(x_j)}} = \left(\frac{a_i}{a_j}\right)^{\sigma} \left(\frac{u'_i(x_i)}{u'_j(x_j)}\right)^{(\sigma-1)} \left(\frac{\bar{u}_i - d_i}{\bar{u}_j - d_j}\right)^{(\sigma-1)^2}$$

And: $$\frac{u_i(x_i) - d_i}{u_j(x_j) - d_j} = \left(\frac{a_i}{a_j}\right)^{\sigma} \left(\frac{u'_i(x_i)}{u'_j(x_j)}\right)^{\sigma} \left(\frac{\bar{u}_i - d_i}{\bar{u}_j - d_j}\right)^{(\sigma-1)^2}$$

Obviously, if $\sigma = 0$, we recover the K-S solution. If $\sigma = 1$, the Nash allocation. But now, for σ above 1, the right hand-side expressions are expected to rise with $\frac{\bar{u}_i - d_i}{\bar{u}_j - d_j}$.

. n-person generalizations of the forms above are straight-forward and we will not burden the exposition with them.

5. Generalizations Weighing the "Down-and-Up" Distance Ratio.

. Few forms of the objective function have contemplated the ideal utility of the players. An alternative formulation of the Nash maximand would not only weight the distances of the utility obtained with the transaction but also – and in symmetric relation to the former – the distances of the obtained utilities to the ideal ones:

$$\text{(35)} \qquad u^{NK} = \underset{u \in G}{\text{Arg Max}} \left[\frac{u_1 - d_1}{\bar{u}_1 - u_1}\right]^{\gamma_1} \left[\frac{u_2 - d_2}{\bar{u}_2 - u_2}\right]^{\gamma_2}$$

The structure of the maximand is still of the Cobb-Douglas type; now, each factor is not the distance towards the *status quo* – the Nash form -, but the ratio of that distance, which each individual would like to maximize, to the distance to the ideal utility attained – that the player would care to minimize. Of course, the function would be ill-defined for negative values of any (one…) of those distances – the relevant G is assumed or restricted to be contained in the rectangle with southwest and northeast vertices in points (d_1, d_2) and ($\bar{u}_1$, $\bar{u}_2$) respectively.

If the function (35) is increasing on either u_i for values in the range $d_i < u_i < \bar{u}_i$, however, quasi-concavity – convex indifference curves – is not guaranteed for all pairs (u_1, u_2), even if within that range. We display below some typical indifference curves, defined for (d_1, d_2) = (0.5, 0.5) and ($\bar{u}_1$, $\bar{u}_2$) = (1.5, 1.5) and equal bargaining strength of the players, $\gamma_1 = \gamma_2 = 0.5$:

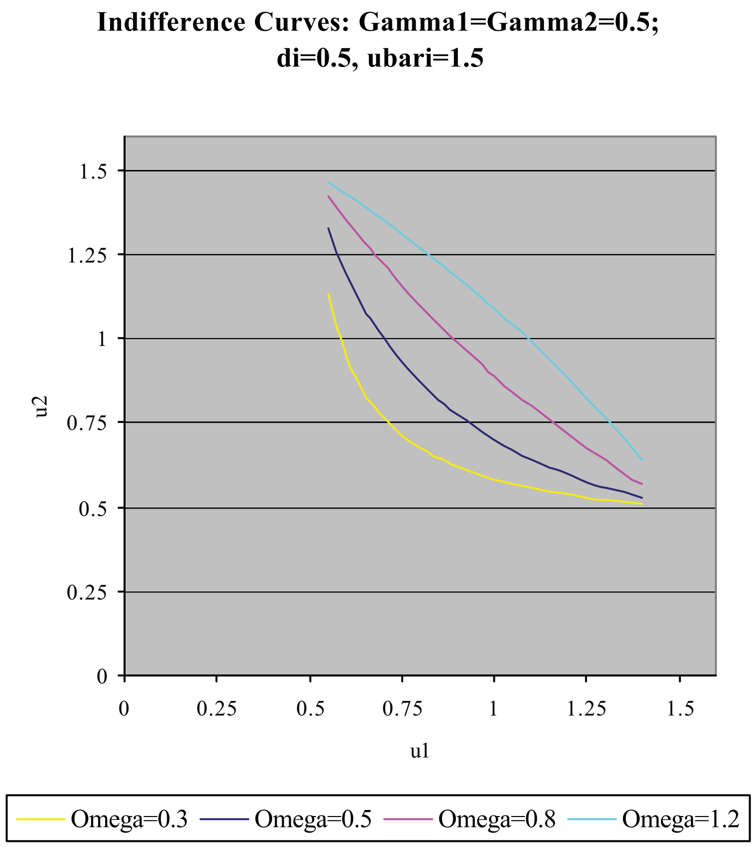

Figure 2. Indifference Curves, $\gamma_1 = \gamma_2 = 0.5$

And unequal bargaining strength of the players, $\gamma_1 = \dfrac{1}{3}; \gamma_2 = \dfrac{2}{3}$:

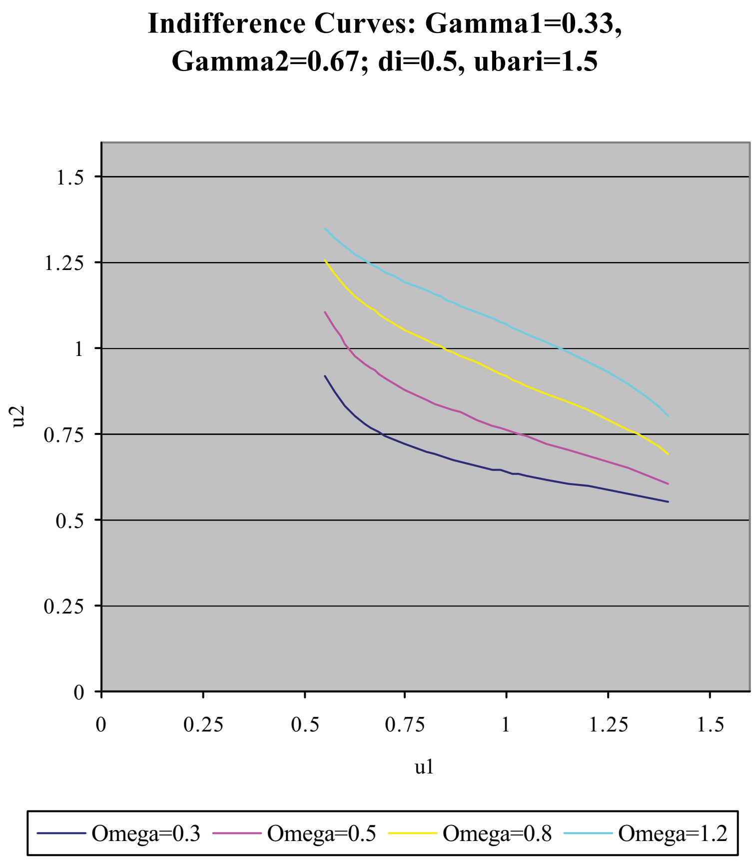

Figure 3. Indifference Curves, $\gamma_1 = 0.33$; $\gamma_2 = 0.67$

As we move, from (d_1, d_2) to the southwest, the indifference curves become less convex and even concave in some utilities range. Unbalance bargaining strength of the players – as expected - renders the curves asymmetric with respect to the bisectrix of the square defined by the points $(d_1, d_2) = (0.5, 0.5)$ and $(\bar{u}_1, \bar{u}_2) = (1.5, 1.5)$.

For the functional form (35), the MRS_{u_1,u_2} is given by:

$$(36) \qquad MRS_{u_1,u_2} = \frac{\frac{\partial\Omega}{\partial u_1}}{\frac{\partial\Omega}{\partial u_2}} = \frac{\gamma_1}{\gamma_2}\,\frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}\,\frac{u_2 - d_2}{u_1 - d_1}\,\frac{\bar{u}_2 - u_2}{\bar{u}_1 - u_1}$$

$MRS_{u_1,u_2} = 1$ originates: $$\frac{u_2 - d_2}{u_1 - d_1}\,\frac{\bar{u}_2 - u_2}{\bar{u}_1 - u_1} = \frac{\gamma_2}{\gamma_1}\,\frac{\bar{u}_2 - d_2}{\bar{u}_1 - d_1}$$

It can be shown that for a point (u_1, u_2) obeying the F.O.C. for a maximum of the function subject to the restriction $u_1 + u_2 = \bar{U}$, S.O.C. (for a local maximum) will be satisfied iff:

$$\frac{2u_1 - (\bar{u}_1 + d_1)}{\gamma_1(\bar{u}_1 - d_1)} + \frac{2u_2 - (\bar{u}_2 + d_2)}{\gamma_2(\bar{u}_2 - d_2)} < 0$$

If (but not only if) $u_i < \frac{\bar{u}_i + d_i}{2}$, i = 1,2, or $u_i - d_i < \frac{\bar{u}_i - d_i}{2}$: the utility gain obtained with the transaction has to be less than half the "ideal gain".

That is, for an interior solution to correspond to a global maximum, points with co - ordinates $\bar{u}_1$ and $\bar{u}_2$ may be required to lie outside the relevant G. They may still correspond to ideal utilities, but, say, defined in such a way that $\bar{u}_i$ is possible for i only if the other player, j, gets less than d_j.

With the proposed functiona l form, we cannot avoid corner solutions – which imply one of the players obtaining his ideal utility; depending on how high these ideals are defined, such corner outcomes may or not possess distributional justice. Nevertheless, the higher they are, relative to the possibility set, the less likely we expect such corner solutions to arise.

An alternative – complementary - form of the objective function, which can be expected to generate the same properties for interior solutions, eventually appropriate when $\frac{\bar{u}_i + d_i}{2}$ belongs to G is:

$$u^{NK} = \underset{u \geqslant G \text{ (and } u \in G)}{\text{Arg Min}} \left[\frac{u_1 - d_1}{\bar{u}_1 - u_1}\right]^{\gamma_1} \left[\frac{u_2 - d_2}{\bar{u}_2 - u_2}\right]^{\gamma_2}$$

where u ⩾ G stands for u is to the northeast or on the northeastern boundary of set G - in space (u_1, u_2). It is possible that a form of the type:

$$(37) \qquad u^{NK} = \underset{u \in G \text{ and } u \geqslant G}{\text{Arg Min}} \left| \left[\frac{u_1 - d_1}{\bar{u}_1 - u_1}\right]^{\gamma_1} \left[\frac{u_2 - d_2}{\bar{u}_2 - u_2}\right]^{\gamma_2} - 1 \right|$$

will encompass both cases: to chose the split more to the northeast of set G that minimizes the (absolute value of the) distance of the geometri c mean (if $\gamma_1 + \gamma_2 = 1$) of the two ratios to 1 – the value of $\left[\frac{u_1 - d_1}{\bar{u}_1 - u_1}\right]^{\gamma_1} \left[\frac{u_2 - d_2}{\bar{u}_2 - u_2}\right]^{\gamma_2}$ when each factor is 1, that is, evaluated in the mid-point between the status quo and the ideal utility, i.e., at $u_i = \frac{\bar{u}_i + d_i}{2}$, i = 1,2.

. Define *pessimism* at a given allocation x_i, $p_i(x_i)$, as

(38) $$p_i(x_i) = \frac{\underset{u \in G}{Max} u_i - u_i(x_i)}{u_i'(x_i)} = \frac{\bar{u}_i - u_i(x_i)}{u_i'(x_i)}$$

It measures how the possible incremental utility relative to what i has already obtained is exchanged per marginal utility, i.e., per infinitesimal incremental utility per unit of x_i. The lower $p_i(x)$ achieved, in better position an individual is in what concerns the possibility set, and the more is he able/expected – the less he would need to reach the ideal - to exchange potential gains of incremental utility relative to the amount measured by marginal utility. Opportunism is the sum of fear of disagreement and pessimism – or, fear of disagreement is the difference between opportunism and pessimism:

(39) $$o_i(x_i) = f_i(x_i) + p_i(x_i)$$

The F.O.C. for a maximum under a split of X, with x_i being the argument the individual i's utility function:

(40) $$\frac{u_2(x_2)-d_2}{u_1(x_1)-d_1}\,\frac{\bar{u}_2 - u_2(x_2)}{\bar{u}_1 - u_1(x_1)}\,\frac{u_1'(x_1)}{u_2'(x_2)} = \frac{\gamma_2}{\gamma_1}\,\frac{\bar{u}_2 - d_2}{\bar{u}_1 - d_1}$$

It can be identified as originating:

(41) $$\frac{f_i(x_i)}{f_j(x_j)}\,\frac{p_i(x_i)}{p_j(x_j)} = \frac{\gamma_i}{\gamma_j}\,\frac{o_i(x_i)}{o_j(x_j)}$$

The geometric mean of the relative fear of disagreement and relative pessimism of the two individuals is going to be equal to the geometric mean of their relative bargaining strengths and their opportunism.

Proposition 5: A maximand of form (35) can be rationalized by requiring axiomatically that in the optimal solution the geometric mean of the ratio between fear of disagreement between any two players and of the corresponding relative degree of pessimism equates the geometric mean of the ratio between their degrees of opportunism and their relative bargaining strengths.

. A natural generalization applies the CES generic form to the factors of the previous specification (35):

$$(42) \qquad u^W = \underset{u \in G}{\text{Arg Max}} \left[a_1 \left(c_1 \frac{u_1 - d_1}{\bar{u}_1 - u_1}\right)^{\frac{\sigma-1}{\sigma}} + a_2 \left(c_2 \frac{u_2 - d_2}{\bar{u}_2 - u_2}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

We display below the typical indifference curves, defined for $(d_1, d_2) = (0.5, 0.5)$ and $(\bar{u}_1, \bar{u}_2) = (1.5, 1.5)$, for the case where $c_1 = c_2 = 1$ and of equal bargaining strength of the players, $a_1 = a_2 = 0.5$, for several values of σ, the elasticity of substitution:

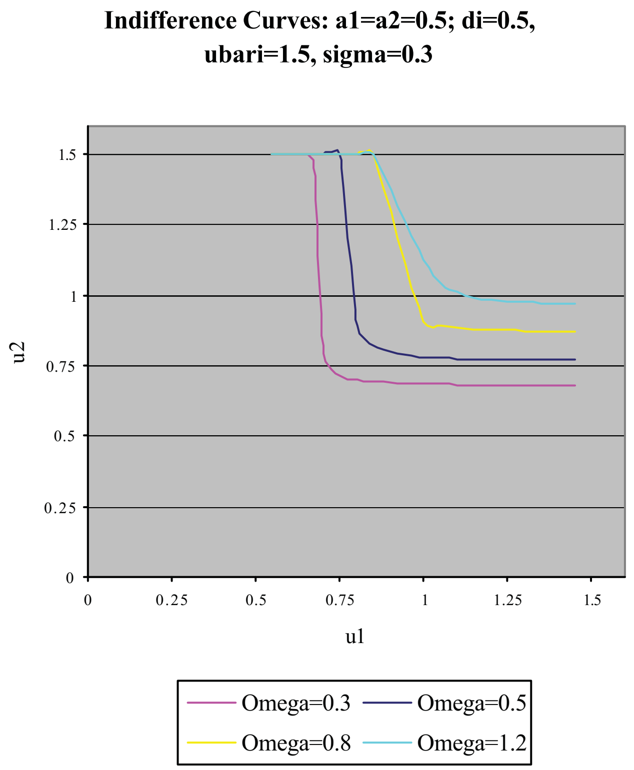

Figure 4. Indifference Curves, $a_1 = a_2 = 0.5$, $\sigma = 0.3$

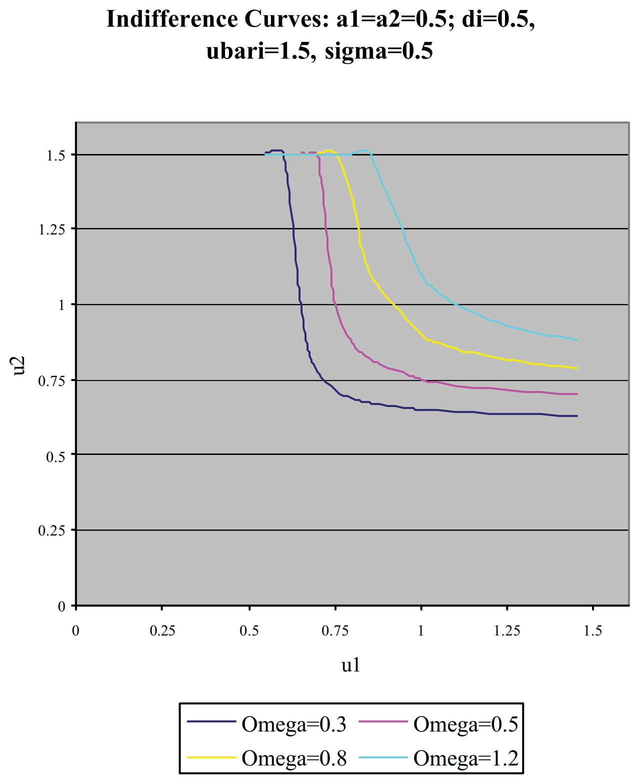

Figure 5. Indifference Curves, $a_1 = a_2 = 0.5$, $\sigma = 0.5$

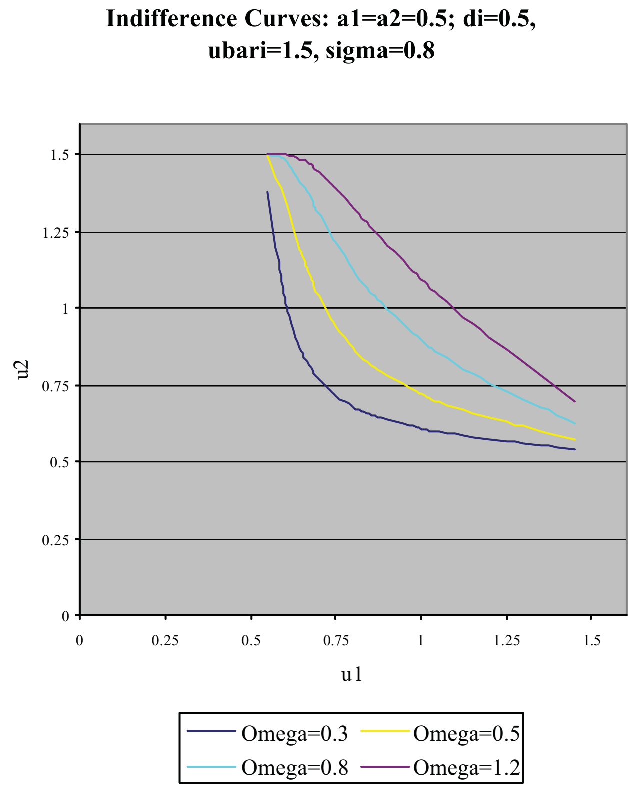

Figure 6. Indifference Curves, $a_1 = a_2 = 0.5$, $\sigma = 0.8$

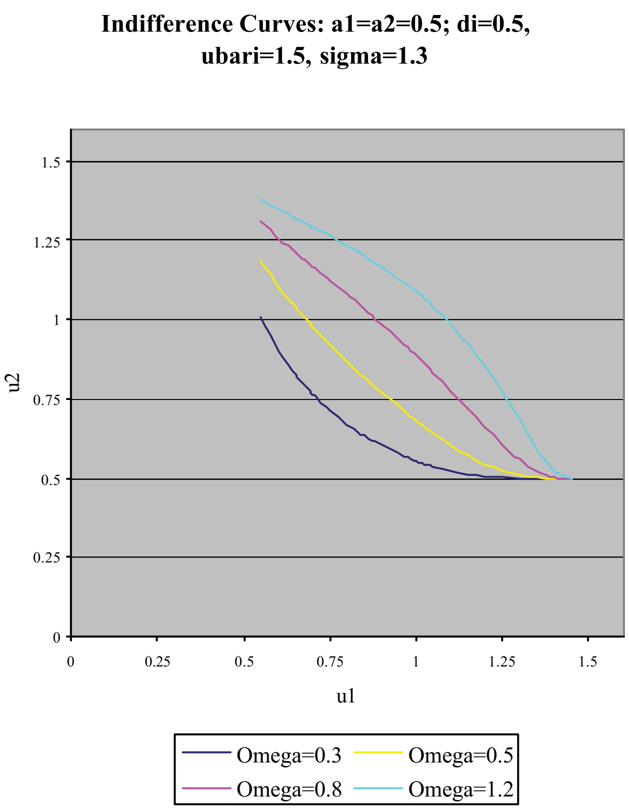

Figure 7. Indifference Curves, $a_1 = a_2 = 0.5$, $\sigma = 1.3$

We reproduce below the same indifference curves under unequal bargaining strength of the players, $a_1 = \frac{1}{3}$; $a_2 = \frac{2}{3}$:

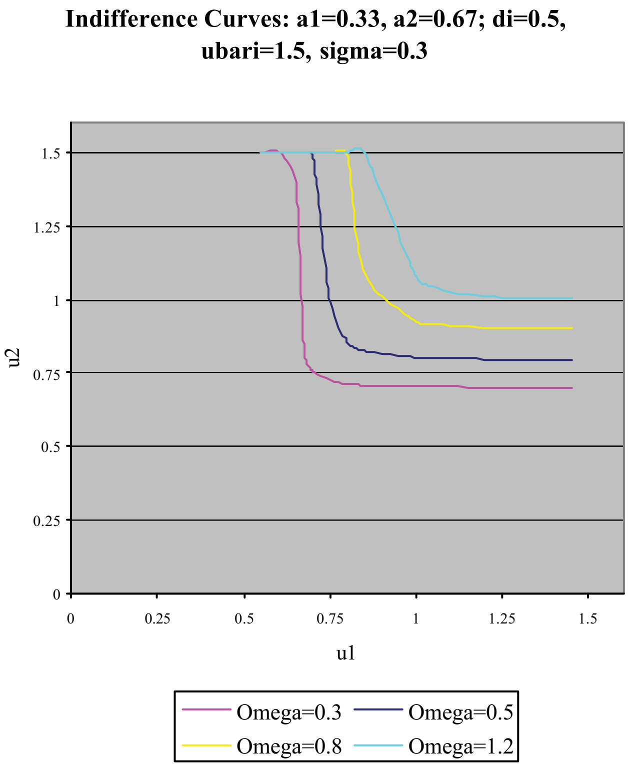

Figure 8. Indifference Curves, $a_1 = 0.33$; $a_2 = 0.67$; $\sigma = 0.3$

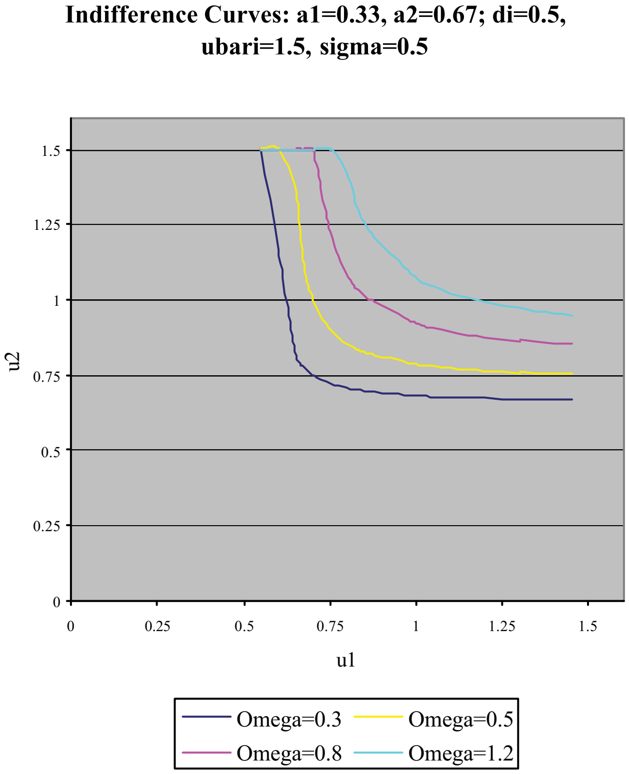

Figure 9. Indifference Curves, $a_1 = 0.33$; $a_2 = 0.67$; $\sigma = 0.5$

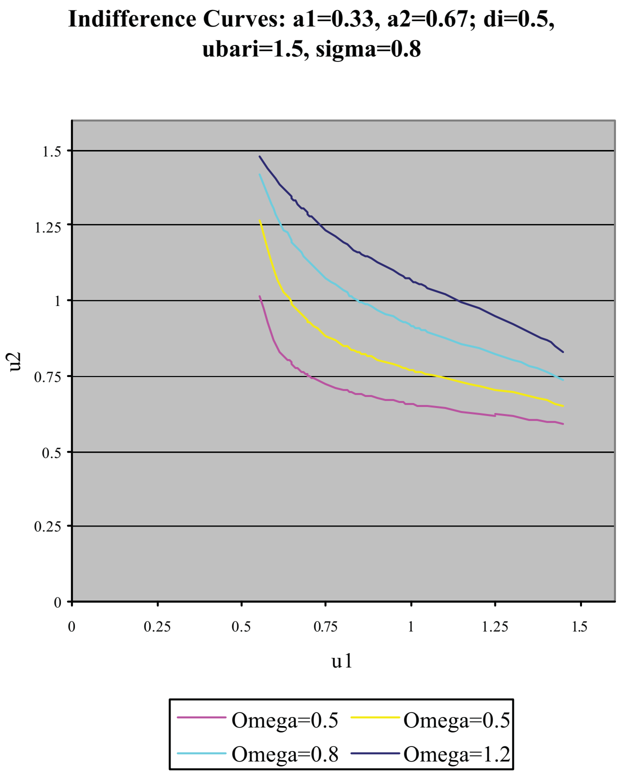

Figure 10. Indifference Curves, $a_1 = 0.33$; $a_2 = 0.67$; $\sigma = 0.8$

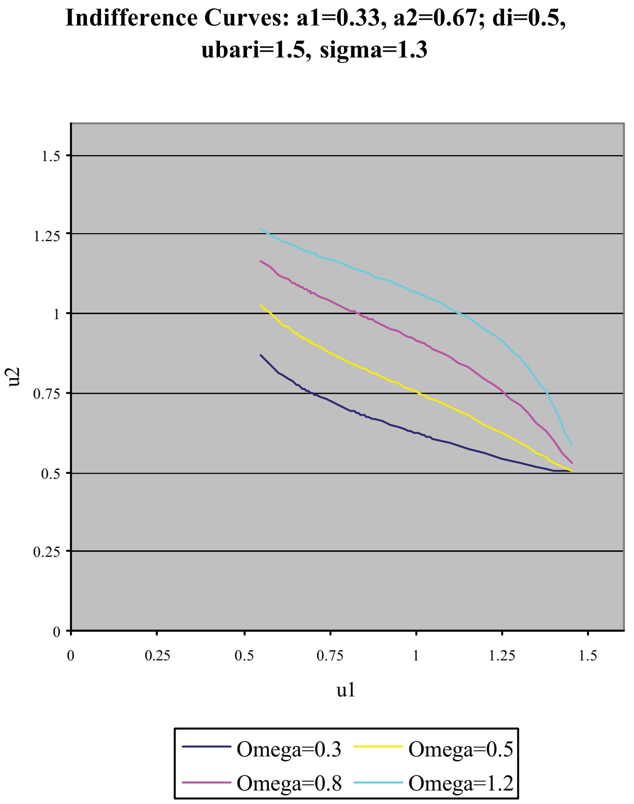

Figure 11. Indifference Curves, $a_1 = 0.33$; $a_2 = 0.67$; $\sigma = 1.3$

In general, similar comments as were made for the Cobb -Douglas form indifference curves apply: there is no guarantee of quasi -concavity of the function in the elementary arguments. Now, as σ rises, the indifference curves become smoother and less convex – as it decreases, we approximate similar properties of a Leontief technology.

The MRS_{u_1,u_2} is given by:

$$(43) \qquad \mathrm{MRS}_{u_1,u_2} = \frac{\frac{\partial\Omega}{\partial u_1}}{\frac{\partial\Omega}{\partial u_2}} = \left[\frac{c_1}{c_2}\right]^{\frac{\sigma-1}{\sigma}} \frac{a_1}{a_2} \frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2} \left[\frac{u_2 - d_2}{u_1 - d_1}\right]^{\frac{1}{\sigma}} \left[\frac{\bar{u}_2 - u_2}{\bar{u}_1 - u_1}\right]^{\frac{2\sigma-1}{\sigma}}$$

$\mathrm{MRS}_{u_1,u_2} = 1$ originates: $$\left[\frac{u_1 - d_1}{u_2 - d_2}\right]^{\frac{1}{\sigma}} \left[\frac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}\right]^{\frac{2\sigma-1}{\sigma}} = \left[\frac{c_1}{c_2}\right]^{\frac{\sigma-1}{\sigma}} \frac{a_1}{a_2} \frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}$$

The right hand -side of the expression is constant. The left hand -side increases with $\frac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}$ iff σ ≥ 0.5; otherwise it decreases. Noticeably, and as occurred for form (35), $\bar{u}_i - d_i$'s show up in the condition without the need to specifically contemplating them as parameters of the underlying maximand.

Making the exponents of the left hand -side factors of the previous expression correspond to the weights of a geometric mean, with positive weights for the case of σ ≥ 0.5:

$$(44) \qquad \left[\frac{u_1 - d_1}{u_2 - d_2}\right]^{\frac{1}{2\sigma}} \left[\frac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}\right]^{\frac{2\sigma-1}{2\sigma}} = \left[\frac{c_1}{c_2}\right]^{\frac{\sigma-1}{2\sigma}} \left[\frac{a_1}{a_2}\right]^{\frac{1}{2}} \left[\frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}\right]^{\frac{1}{2}}$$

Consider the case $c_1 = c_2 = 1$; then the weighted geometric mean of the ratio of both relevant distances is equated to the geometric mean of relative bargaining strength, $\frac{a_1}{a_2}$, and of the ratio of the maximal ideal distances of the two players.

To interpret the expression in terms of geometric means with positive weights for σ ≤ 0.5 we can re-write the expression as:

$$(45) \qquad \frac{u_1 - d_1}{u_2 - d_2} = \left[\frac{c_1}{c_2}\right]^{(\sigma-1)} \left[\frac{a_1}{a_2}\right]^{\sigma} \left[\frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}\right]^{\sigma} \left[\frac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}\right]^{(1-2\sigma)}$$

If we consider that $c_i = \dfrac{1}{\overline{u}_i - d_i}$, i=1,2, as in the CES form (23), then the geometric mean of the right hand -side becomes also a weighted mean; (44) becomes:

$$\left[\frac{u_1-d_1}{u_2-d_2}\right]^{\frac{1}{\sigma+1}}\left[\frac{\overline{u}_1-u_1}{\overline{u}_2-u_2}\right]^{\frac{2\sigma-1}{\sigma+1}} = \left[\frac{a_1}{a_2}\right]^{\frac{\sigma}{\sigma+1}}\left[\frac{\overline{u}_1-d_1}{\overline{u}_2-d_2}\right]^{\frac{1}{\sigma+1}}$$

For a reading of positive exponents for $\sigma \le 0.5$, it can be developed as:

$$\frac{u_1-d_1}{u_2-d_2} = \left[\frac{a_1}{a_2}\right]^{\sigma}\frac{\overline{u}_1-d_1}{\overline{u}_2-d_2}\left[\frac{\overline{u}_1-u_1}{\overline{u}_2-u_2}\right]^{(1-2\sigma)}$$

. For a split problem on the arguments of the u_i's:

$$(46)\quad \left[\frac{u_i(x_i)-d_i}{u_j(x_j)-d_j}\right]^{\frac{1}{2\sigma}}\left[\frac{\overline{u}_i-u_i(x_i)}{\overline{u}_j-u_j(x_j)}\right]^{\frac{2\sigma-1}{2\sigma}} = \left[\frac{c_i}{c_j}\right]^{\frac{\sigma-1}{2\sigma}}\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\left[\frac{\overline{u}_i-d_i}{\overline{u}_j-d_j}\right]^{\frac{1}{2}}\left[\frac{u_i'(x_i)}{u_j'(x_j)}\right]^{\frac{1}{2}}$$

Multiplying both sides of the expression (46) by $\dfrac{u_j'(x_j)}{u_i'(x_i)}$, we can correspond:

$$(47)\quad \left[\frac{f_i(x_i)}{f_j(x_j)}\right]^{\frac{1}{2\sigma}}\left[\frac{p_i(x_i)}{p_j(x_j)}\right]^{\frac{2\sigma-1}{2\sigma}} = \left[\frac{c_i}{c_j}\right]^{\frac{\sigma-1}{2\sigma}}\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\left[\frac{o_i(x_i)}{o_j(x_j)}\right]^{\frac{1}{2}}$$

Admit $c_1 = c_2 = 1$; then, the weighted geometric mean of relative fear of disagreement and relative pessimism – with larger weight given to the former the smaller σ, and with positive weights only if $\sigma \ge 0.5$ - is equated to the geometric mean of relative bargaining strength and relative opportunism.

Alternatively, for $\sigma \le 0.5$:

$$(48)\quad \frac{f_i(x_i)}{f_j(x_j)} = \left[\frac{c_i}{c_j}\right]^{(\sigma-1)}\left[\frac{a_i}{a_j}\right]^{\sigma}\left[\frac{o_i(x_i)}{o_j(x_j)}\right]^{\sigma}\left[\frac{p_i(x_i)}{p_j(x_j)}\right]^{1-2\sigma}$$

With $c_1 = c_2 = 1$, for $\sigma \le 0.5$, relative fear of disagreement is equated to the geometric mean of relative bargaining strength, relative opportunism and relative pessimism of the players.

If we consider instead that $c_i = \dfrac{1}{\overline{u}_i - d_i}$, i = 1,2:

$$\left[\frac{u_1-d_1}{u_2-d_2}\right]^{\frac{1}{\sigma+1}}\left[\frac{\bar{u}_1-u_1}{\bar{u}_2-u_2}\right]^{\frac{2\sigma-1}{\sigma+1}} = \left[\frac{a_1}{a_2}\right]^{\frac{\sigma}{\sigma+1}}\left[\frac{\bar{u}_1-d_1}{\bar{u}_2-d_2}\right]^{\frac{1}{\sigma+1}}\left[\frac{u_i'(x_i)}{u_j'(x_j)}\right]^{\frac{\sigma}{\sigma+1}}$$

. Interesting cases are, thus:

1) $\sigma = 1$; $c_1 = c_2 = 1$. We recover form (35), with the a_i's taking the place of the γ_i's.

2) $\sigma = 0.5$. Then, $MRS_{u_1,u_2} = 1$ implies

(49) $$\frac{u_1-d_1}{u_2-d_2} = \left[\frac{c_1}{c_2}\right]^{-\frac{1}{2}}\left[\frac{a_1}{a_2}\right]^{\frac{1}{2}}\left[\frac{\bar{u}_1-d_1}{\bar{u}_2-d_2}\right]^{\frac{1}{2}}$$

The split of X originates: $\dfrac{u_i(x_i)-d_i}{u_j(x_j)-d_j} = \left[\dfrac{c_i}{c_j}\right]^{-\frac{1}{2}}\left[\dfrac{a_i}{a_j}\right]^{\frac{1}{2}}\left[\dfrac{\bar{u}_i-d_i}{\bar{u}_j-d_j}\right]^{\frac{1}{2}}\left[\dfrac{u_i'(x_i)}{u_i'(x_j)}\right]^{\frac{1}{2}}$ or

(50) $$\frac{f_i(x_i)}{f_j(x_j)} = \left[\frac{c_i}{c_j}\right]^{-\frac{1}{2}}\left[\frac{a_i}{a_j}\right]^{\frac{1}{2}}\left[\frac{o_i(x_i)}{o_i(x_j)}\right]^{\frac{1}{2}}$$

For the case where $c_1 = c_2 = 1$, we recover the particular case of (30) for an elasticity of substitution of also 0.5.

For the case where $c_i = \dfrac{1}{\bar{u}_i-d_i}$, i = 1,2, $MRS_{u_1,u_2} = 1$ implies:

$$\frac{u_1-d_1}{u_2-d_2} = \left[\frac{a_1}{a_2}\right]^{\frac{1}{2}}\frac{\bar{u}_1-d_1}{\bar{u}_2-d_2}$$

and the split of X problem: $\dfrac{u_i(x_i)-d_i}{u_j(x_j)-d_j} = \left[\dfrac{a_i}{a_j}\right]^{\frac{1}{2}}\dfrac{\bar{u}_i-d_i}{\bar{u}_j-d_j}\left[\dfrac{u_i'(x_i)}{u_i'(x_j)}\right]^{\frac{1}{2}}$

3) $\sigma = 0$. $MRS_{u_1,u_2} = 1$, and likewise the split of X, originate:

(51) $$\frac{u_1-d_1}{u_2-d_2} = \left[\frac{c_1}{c_2}\right]^{-1}\frac{\bar{u}_1-u_1}{\bar{u}_2-u_2}$$

Then, we would reproduce a Leontief maximand over the distance ratios:

(52) $$u^W = \underset{u \in G}{\text{Arg Max}}\ \text{Min}\left(c_1\frac{u_1-d_1}{\bar{u}_1-u_1},\ c_2\frac{u_2-d_2}{\bar{u}_2-u_2}\right)$$

If $c_1 = c_2 = 1$: $\dfrac{u_1 - d_1}{u_2 - d_2} = \dfrac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}$ or equality between the two individuals "down-and-up" distance ratios: $\dfrac{u_1 - d_1}{\bar{u}_1 - u_1} = \dfrac{u_2 - d_2}{\bar{u}_2 - u_2}$.

If $c_i = \dfrac{1}{\bar{u}_i - d_i}$, i=1,2, the ratio of those ratios will be proportional to the relative ideal incremental utilities of the players: $\dfrac{u_1 - d_1}{u_2 - d_2} = \dfrac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2} \dfrac{\bar{u}_1 - u_1}{\bar{u}_2 - u_2}$

Proposition 6: A maximand of form (41) with $c_1 = c_2 = 1$, a generalization of (35), can be rationalized by requiring axiomatically that in the optimal solution

6.1. for $\sigma \geq 0.5$, that the *weighted* geometric mean of the ratio between fear of disagreement between any two players and of the corresponding relative degree of pessimism – with the relative weight to the former inversely related to the elasticity of substitution parameter - equates the geometric mean of the ratio between their degrees of opportunism and their relative bargaining strengths.

6.2. for $\sigma \leq 0.5$, that fear of disagreement is equated to the *weighted* geometric mean of relative bargaining strength, relative opportunism and relative pessimism of the players – with the weight given to the latter tending to 0 as σ tends to 0.5.

Finally, comparison of forms (35) and (42) with (23) can now be forwarded: opportunism always reinforces bargaining strength in the solutions implied by the forms of the current sub-section; under (23), only for an elasticity smaller than 1 is that relation obtained.

Simultaneously, pessimism does not show up in (23) – it reinforces the effect of opportunism (and bargaining strength) in (35) and (42) for low values of σ - $\sigma \leq 0.5$ -, it counteracts for high values; as σ rises, indifference curves become more convex – and, hence, corner solutions – potentially, less egalitarian - more likely to emerge.

. In line with (37), an objective function of the form:

$$(53)\quad u^{NK} = \text{Arg Min}\ \left| \left[a_1 \left(c_1 \frac{u_1 - d_1}{\bar{u}_1 - u_1}\right)^{\frac{\sigma-1}{\sigma}} + a_2 \left(c_2 \frac{u_2 - d_2}{\bar{u}_2 - u_2}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - \left(a_1\, c_1^{\frac{\sigma-1}{\sigma}} + a_2\, c_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right|$$

$u \in G$ and $u \geqslant G$

will reproduce the interior F.O.C. with the same displayed properties.

Conclusion

The maximum utility each player may possibly achieve in a given negotiation may be a more important reference point to the players than conventional cooperative games benchmark solutions imply. This research presented several generalizations of the latter, under which the equilibrium possesses such mathematical property. In some, a direct reference to a potential unilateral appropriation – out of the bargaining table – of the surplus being bargained over was incorporated.

It was immediate to conclude conditions for likelihood of bargaining itself being observed: a low probability of unilateral victory in appropriation of the total surplus, a low "status quo" perspective, and a high total surplus to be split.

Additionally, generalizations of both Nash and Kalai-Smorodinsky solutions were devised, namely, CES maximands, one defined over incremental utilities; another on relative "down and up" utility distances, justified on analogs to risk-aversion measures applicable in the presence of large risks.

In the former, relative fear of disagreement is equated to a weighted geometric mean of relative "financial bargaining strength" – defined with reference to the special case originating the Nash form - and relative opportunistic propensity of any two players. A higher elasticity of substitution parameter of the CES can be seen to accommodate more or less egalitarian distribution of the bargaining game according to the relative financial strength of the players.

In the latter, for levels of the elasticity of substitution larger than 0.5, the – weighted or not - geometric mean of relative fear of disagreement and relative pessimism of any two players balances with the geometric mean of their relative bargaining strength and relative opportunistic propensity; for lower levels of the elasticity, relative fear of disagreement equates to weighted geometric mean of relative bargaining strength, relative opportunism and relative pessimism. For this form, very high elasticity of substitution parameter of the CES – generating concave indifference curves of the underlying pseudo-social indifference curve in the utilities space - may lead to more unbalance distributions of the bargaining gains, and to corner solutions of that distribution.

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