Splitting Games: Nash Equilibrium and the Optimisation Problem

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ABSTRACT

This research states the stylised n players' splitting problem as a mathematical programme, relying on definitions of the values of the game and problem stationarity to generate tractable reduced forms, and derives the known solutions after pertaining first-order conditions. Boundary constraints are introduced. Distinction between FOC's of optimising behavior and equilibrium fitness is provided. Finally, the formal proof of the internal insufficiency of the usual approach to determine the equilibrium is advanced, and the imposing additional conditions – affecting cross multipliers – required for model solving forwarded. Two types of protocols were staged: alternate offers – Rubinstein's like – and synchronised ones.

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1 INTRODUCTION

1.1 Background

Rubinstein's (1982) structure has become a major reference in game theory and wage bargaining literature, possessing in its most well-known form the agreeable characteristic of generating – under perfect information, rational players with positive discount rates, and a realistic bargaining protocol – an immediate settlement and a unique equilibrium with no time loss, illustrating both the first-mover and patience advantages. Moreover, it provided, after Binmore, Rubinstein and Wolinsky (1986), a rationale for the cooperative solution implied by the widely accepted Nash (1950, 1953) maximand. Martins (2006) proposed generalisations of its infinite horizon solution to games involving more than two players. It is the purpose of this research to suggest the pertaining solutions as stemming from equilibrium of first-order conditions of conventional (quasi-)static optimisation programmes.

In all scenarios, at stake is the split of an infinite flow of benefits, periodically available at subsequent, equally distant, discrete points in time. For simplicity, it is assumed that only stationary divisions of the cake are contractually acceptable, and enforceable, once agreed upon, ad infinitum – a context akin to wage bargaining, but also realistic for other settings, namely rental - tenancy and leasing – agreements, and barter of capital or durable goods. Each player makes a proposition on the division of the cake among all the players and only one proposal is heard each period; if refused, each agent enjoys in the period an exogenous alternative specific to the opponent making offers, and the bargaining re-initiates – continues - next period.

Firstly, we advance alternate offers protocols: at a player's turn to "make a move", he can chose not to make any, enjoying a pay-off different from the one he gets when refusing an offer, and wait for next period's decision of the opponent concerning the same choice. Only the player that is going to make the immediate offer – i.e., the first player at each point in time - is known, and each player except the one that is making the offer can end up making one with equal probability next period.

We allow the players to decide whether to negotiate or not, and also to play mixed (i.e., random) strategies. These are known to exist for

familiar bargaining games, even if not necessarily called for to assure equilibrium.

Simultaneous bargaining has been studied in the literature to obviate the dependence of Rubinstein's results on the order and timing of offers. Usually, it is staged in a sequential set-up where time is assumed to be continuous and minimum delays between offers to exist - see Perry and Reny (1993) and Sákovics (1993). Instead, we keep the discrete time and forward the notion of "matching" or synchronous equilibrium, defined relying on each player using a mixed (probabilistic) strategy conditional on (and statistically independent of) the other player's action – as in Martins (2004 and 2006).

The individuals' problems, even if dynamic in nature, exhibit an obvious stationarity: invariably, only a fixed number of alternative situations are (recurrently) possible at any point in time during the game (or while the game does not finish). This implies the existence of a static mathematical representation of the equilibrium - adjusted to account for the proponent's rotation in the case of alternate offers. Individuals maximise the value of the game at each point in time, subject to – under a Nash perspective – the decisions that were/are/will be taken by opponents – and by himself with alternate offers; equilibrium derives from mutual agreement. Game theory does not usually resort to Lagrangean methods – Khun-Tucker conditions - and equilibrating constraints to generate solutions; we therefore inquire why, and if – in the reduced problems – some qualification of the usual conditions can generate the solutions proposed in the literature.

The exposition proceeds as follows: notation is forwarded in section 1 and implications of solution stationarity for the dynamic programmes deducted in section 2. Alternate offers are staged in section 3: starting by the players' problems, then deriving FOC and finally justifying the Nash equilibrium. In section 4, following the same steps, properties of simultaneous, yet sequential, equilibria are inspected. The exposition ends with some concluding remarks.

1.2 Notation[2]

A "pie" of fixed size, normalised to 1, is made available to the n individuals every period; each player has per period utility function – a discrete, well-behaved, "felicity" function - $u_i(z)$, with z denoting the

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[2] We mainly reproduce Martins' (2006) notation.

share obtained by i, which he discounts at factor δ_i – maximising accumulated discounted felicity. Unlike for other parameters, a superscript k on δ_i, i.e., δ_i^k, denotes the k-th power (of δ_i).

Each player is responsible for a proposition of the division of the cake, $x^j = (x_1^j, x_2^j,..., x_i^j,..., x_n^j)$ where x_i^j is the share of player i proposed by player j and therefore:

(1.1) $\sum_{i=1}^{n} x_i^j = 1$, j = 1, 2,..., n

Of course, protocols differ according to the allocation of j's turn to make a proposition.

Each player holds "veto" rights over an agreement – ruling out benefits from coalition-seeking. A (therefore unanimous) agreement on the share of the pie accruing to each player is binding for eternity:[3] if a settlement is reached about the split of the pie for a particular period, the same split will hold forever.[4]

Also, when an individual, j, makes an offer – proposal –, either everybody accepts it and the split is settled, with player i, i = 1,2,...,n, getting accumulated discounted felicity from his perpetual share:

(1.2) $\dfrac{u_i(x_i^j)}{1 - \delta_i}$

Or it is rejected – someone rejects it; in this case, the current pie is lost and d_i^j is the periodic felicity accruing to player i – d_i^j may be $u_i(0)$, or a better alternative exogenously available to him after rejecting j's offer. s_i^j - i≠j - is the probability with which player i rejects j's offer.

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[3] See Manzini (1998), for a survey of similar structures and results, including finite horizon games. Also, Busch and Wen (1995), Muthoo (1995) and Muthoo (1999). Yet, these contracts can sometimes be converted – see Martins (2004) – into a single "pie" division one.

[4] This condition/assumption restricts the relevant strategies to the players to be stationary in the long-run.

d_i^i, the periodic alternative accruing to i if his offer is rejected, is always assumed a low-value option, and enjoys a different status than d_i^j, j≠i.

After a rejection, the "haggling" then reinitiates next period with the player making the offer being determined by the game protocol.

r_i is the probability with which player i makes an offer when he is the one so appointed. Pure strategies with respect to it arise when players decide $r_i = 1$. We assume a player gets d_i in the period if he does not make an offer, and that everyone else, j≠i, simultaneously gets d_j – player i gets periodic felicity d_i if there is no offer exchange in the period. It may differ from the d_i^j's for j≠i, and from d_i^i. Of course, for games in pure strategies, d_i is a bound, but it does not influence (fully) interior solutions. For mixed strategies to emerge, d_i may not coincide with the periodic alternative available for player i outside the game (the latter may then have to be much lower for the solutions advanced in the text to hold).

With alternate offers, each period, one and only player is exogenously assigned the right to make (or not, but then time elapses without the game ending) the periodic offer. V_i^j is then going to denote the value of the game for player i at the point at which j is supposed to make an offer.

With simultaneous offers, V_i is the value of the game for player i at the beginning of the game, as at any point in time while the game is running – has not finished in an infinite term settlement.

2 DYNAMIC OPTIMISATION UNDER STATIONARITY

Individuals' problems are dynamic programming structures. We have no state variables, yet, decisions in one period affect others. Consider a simple (single-person, infinite horizon) problem such that one must decide x_i^t, t = 1,2,…, which affects directly a function that i maximises at time t, $V_i^t = g_i(x_i^t, V_i^{t+1}*)$, knowing that $V_i^{t+1}*$ obeys $V_i^{t+1} =$

$g_i(x_i^{t+1}, V_i^{t+2}*)$ and will be maximised at time t+1 according to the same principles. At (any) time t, i solves:

(2.1) $V_i^{t*} = \underset{\substack{V_i^t, V_i^{t+1}, V_i^{t+2},...; \\ x_i^t, x_i^{t+1}, x_i^{t+2},...;}}{\text{Max}} V_i^t$

(2.2) s.t.: $V_i^{t+k} = g_i(x_i^{t+k}, V_i^{t+k+1})$, k = 0,1,2...

and $V_i^{t+k} = V_i^{t+k}*$, k = 1,2...

$x_i^{t+k} = x_i^{t+k}*$, k = 1,2...

i.e., knowing that V_i^{t+k} and x_i^{t+k} will be consistent with an optimal choice in those later periods. Given the stationarity of the problem, $V_i^{t+k*} = V_i^*$ and $x_i^{t+k} = x_i^*$ - whatever optimal at time t, will also be it at t+k and vice-versa -, which can therefore be replaced, simplifying the structure to:

(2.3) $\underset{V_i; x_i}{\text{Max}} V_i$

(2.4) s.t.: $V_i = g_i(x_i, V_i)$

We will resort to the argument to suggest simplifying maximands in both type of protocols. For synchronised offers, it will be sufficient to generate a consistent game between n players – whose problems interact, but where recurrence of optimal decisions is expected; with alternate offers, a further generalisation – but in the same spirit –, isolating the n possible stationary states for each player, provides mathematical structures also easier to deal with.

3 ALTERNATE OFFERS

3.1 The Player's Problem

A player can either be appointed to make an offer in each period, or not. If he is and negotiations break-down, another of the n-1 players will take that role in the following round.

If i is making the offer, he will solve the problem:

(3.1) $\max\limits_{\substack{V_1^1,...,V_1^i,...,V_1^n;...;\\ V_i^1,...,V_i^i,...,V_i^n;...;\\ V_n^1,...,V_n^i,...,V_n^n;\\ r_i;\ x_1^i,...,x_i^i,...,x_n^i;\\ s_i^1,...,s_i^{i-1},s_i^{i+1},...,s_i^n}} V_i^i$, s.t.:

$$(3.2)\ V_l^m = r_m\ [\ \prod_{\substack{k=1\\k\neq m}}^n s_k^m\ \frac{u_l(x_l^m)}{1-\delta_l} + (1 - \prod_{\substack{k=1\\k\neq m}}^n s_k^m\ )\ (d_l^{\ m} + \frac{\delta_l}{n-1}\ \sum_{\substack{k=1\\k\neq m}}^n V_l^k\ )]$$

$$+ (1 - r_m)\ (d_l + \frac{\delta_l}{n-1}\ \sum_{\substack{k=1\\k\neq m}}^n V_l^k\ ),\ l,m = 1,2,...,n$$

$$\sum_{l=1}^n x_l^i\ = 1$$

$$x_l^i \geq 0,\ l = 1,2,...,n;$$

$$0 \leq r_i \leq 1;\ 0 \leq s_i^{\ m} \leq\ 1,\ m\neq i,\ m = 1,2,...,n$$

Given r_m, $m\neq i$, $m = 1,2,...,n$;

x_l^m, $m\neq i$, $l,m = 1,2,...,n$;

$s_l^{\ m}$, $l\neq i,m$, $l,m = 1,2,...,n$.

If player j is the one making a proposition, i solves:

(3.3) $\max\limits_{\substack{V_1^1,...,V_1^i,...,V_1^n;...;\\ V_i^1,...,V_i^i,...,V_i^n;...;\\ V_n^1,...,V_n^i,...,V_n^n;\\ r_i;\ x_1^i,...,x_i^i,...,x_n^i;\\ s_i^1,...,s_i^{i-1},s_i^{i+1},...,s_i^n}} V_i^j$

and he is subject to the same conditions.

Given that the two positions intertwine, the same controls – and constraints – are present. And each player i solves n problems: maximising V_i^j, for each j = 1,2,...,n.

The fact that all the V_l^m's appear as controls is instrumental – they show as controls but are constrained by their definitions, known to the

players, stated in (3.2). On the one hand, the n equations defining V_i^m, m = 1,2,...,n, would allow a unique solution for $V_i^j = g()$, where g has arguments other than the V_i^m's – analogously to form (2.4). As now the own decisions affect other players, V_l^m's, l≠i must also be specified.

Independent optimisation, with simultaneous decision of the same controls, is justified by a simple analog to the envelope theorem: we have that each V_i^j depends on control z_l^i and on the optimal V_i^k's, k≠j, k =1,2,...,n (as we will see, V_r^m for r≠i will not affect directly V_i^j). Then, when deciding at the point where an offer from j is being analyzed, i is maximising the present value of the game, knowing that he will also optimise at later dates; he will program the (any at his disposal) control z_l^i in such a way that

$$\frac{dV_i^j}{dz_l^i} = \frac{\partial V_i^j}{\partial z_l^i} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{\partial V_i^j}{\partial V_i^k}$$

$\frac{dV_i^k}{dz_l^i} = 0$. Of course, at time k, $\frac{dV_i^k}{dz_l^i} = 0$ – at the time k is "dealing", l is seen as posterior; but then, $\frac{\partial V_i^j}{\partial z_l^i}$ (as $\frac{\partial V_i^k}{\partial z_l^i}$) must be set to zero. (In general, constrained maximisation will also obey this condition because in an optimal solution $\frac{dL_i^j}{dz_l^i} = \frac{dV_i^j}{dz_l^i}$, where L_i^j denotes the appropriate Lagrangean of the problem associated directly to V_i^j). This implies that we could have stated the two problem controls as the contemporaneous ones only, i.e.,

$$\max_{\substack{V_1^1,...,V_1^i,...,V_1^n;...; \\ V_i^1,...,V_i^i,...,V_i^n;...; \\ V_n^1,...,V_n^i,...,V_n^n; \\ r_i; \, x_1^i,...,x_i^i,...,x_n^i}} V_i^i \quad \text{and} \quad \max_{\substack{V_1^1,...,V_1^i,...,V_1^n;...; \\ V_i^1,...,V_i^i,...,V_i^n;...; \\ V_n^1,...,V_n^i,...,V_n^n; \\ s_i^j}}$$

V_i^j for j≠i – but provided the resulting system would be sufficient for determination.

3.2 First-Order Conditions

Each player's responses – strategies – are going to be consistent – i.e., they will be formed after the compatibilisation of FOC of each of his n optimisation programs.

For each problem j, player i is going to face the Lagrangean:

(3.4)
$$\underset{\substack{V_1^1,...,V_1^i,...,V_1^n;...; \\ V_i^1,...,V_i^i,...,V_i^n;...; \\ V_n^1,...,V_n^i,...,V_n^n; \\ r_i;\ x_1^i,...,x_i^i,...,x_n^i; \\ s_i^1,...,s_i^{i-1},s_i^{i+1},...,s_i^n; \\ (\mu_1^1)_i^j,...,(\mu_1^i)_i^j,...,(\mu_1^n)_i^j;...; \\ (\mu_i^1)_i^j,...,(\mu_i^i)_i^j,...,(\mu_i^n)_i^j;...; \\ (\mu_n^1)_i^j,...,(\mu_n^i)_i^j,...,(\mu_n^n)_i^j; \\ (\lambda)_i^j; \\ (\eta^1)_i^j,...,(\eta^{i-1})_i^j,(\eta^{i+1})_i^j...,(\eta^n)_i^j;(\eta)_i^j}}{\text{Max}} \quad L_i^j = V_i^j + \sum_{m=1}^{n} \sum_{l=1}^{n} (\mu_l^m)_i^j \{- V_l^m + r_m$$

$$[\prod_{\substack{k=1 \\ k\neq m}}^{n} s_k^m \ \frac{u_l(x_l^m)}{1-\delta_l} + (1 - \prod_{\substack{k=1 \\ k\neq m}}^{n} s_k^m) \ (d_l^m + \frac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k\neq m}}^{n} V_l^k)] + (1 - r_m) \ (d_l$$

$$+ \frac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k\neq m}}^{n} V_l^k)\} + (\lambda)_i^j \ (1 - \sum_{l=1}^{n} x_l^i) + (\eta)_i^j \ (1 - r_i) + \sum_{\substack{m=1 \\ m\neq i}}^{n} (\eta^m)_i^j$$

$$(1 - s_i^m)$$

$$x_l^i \geq 0,\ l = 1,2,...,n;$$

$$(\lambda)_i^j \geq 0;\ (\mu_l^m)_i^j \geq 0,\ l,m = 1,2,...,n;$$

$$(\eta)_i^j \ (1 - r_i) = 0,\ r_i \geq 0,\ (\eta)_i^j \geq 0;$$

$$(\eta^m)_i^j \ (1 - s_i^m) = 0,\ s_i^m \geq 0,\ (\eta^m)_i^j \geq 0,\ m\neq i,\ m = 1,2,...,n$$

The optimand is linear in the controls. For a maximum, equality as inequality constraint devices – for application of Khun-Tucker conditions – are added to the constraints and embedded in the Lagrangean, constructed to exhibit non-negative multipliers – to obey SOC, that with a linear maximand, are satisfied with convexity of the constraints.[5]

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[5] Concavity of the definitions of V_l^m in the arguments – that requires concavity of felicity functions in pure strategies, reason why in the Lagrangean their symmetric is in fact introduced: (if we replaced the definition of V_i^j in the maximand, it should be

Then, for solutions with positive values of x_l^i, s_i^j and r_i we have:[6]

(3.5) $\dfrac{\partial L_i^j}{\partial V_i^j} = 1 - (\mu_i^j)_i^j + \displaystyle\sum_{\substack{m=1 \\ m \neq j}}^{n} (\mu_i^m)_i^j \left(1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m\right) \dfrac{\delta_i}{n-1} = 0$

(3.6) $\dfrac{\partial L_i^j}{\partial V_r^s} = -(\mu_r^s)_i^j + \displaystyle\sum_{\substack{m=1 \\ m \neq s}}^{n} (\mu_r^m)_i^j \left(1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m\right) \dfrac{\delta_r}{n-1} = 0,\ r \neq i$ or

$s \neq j,\ r,s = 1,2,\dots,n$

(3.7) $\dfrac{\partial L_i^j}{\partial x_r^i} = (\mu_r^i)_i^j\, r_i \displaystyle\prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i\ \dfrac{u_r{}'(x_r^i)}{1 - \delta_r} - (\lambda)_i^j = 0,\ r = 1,2,\dots,n$

(3.8) $\dfrac{\partial L_i^j}{\partial s_i^s} = \displaystyle\sum_{\substack{m=1 \\ m \neq s}}^{n} \sum_{l=1}^{n} (\mu_l^m)_i^j\, \Bigg\{ r_m \Bigg[\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m\ \dfrac{u_l(x_l^m)}{1 - \delta_l} - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m\ \Big(d_l^m + $

$\dfrac{\delta_l}{n-1} \displaystyle\sum_{\substack{k=1 \\ k \neq m}}^{n} V_l^k \Big) \Bigg] \Bigg\} / s_i^s - (\eta^s)_i^j = 0,\ s \neq i,\ s = 1,2,\dots,n$

(3.9) $\dfrac{\partial L_i^j}{\partial r_i} = \displaystyle\sum_{l=1}^{n} (\mu_l^i)_i^j\, \Bigg\{ \Bigg[\prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i\ \dfrac{u_l(x_l^i)}{1 - \delta_l} + \Big(1 - \prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i\Big) \Big(d_l^i + \dfrac{\delta_l}{n-1} $

$\displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{n} V_l^k \Big) \Bigg] - \Big(d_l + \dfrac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k \neq i}}^{n} V_l^k \Big) \Bigg\} - (\eta)_i^j = 0$

(3.10) $\dfrac{\partial L_i^j}{\partial (\mu_l^m)_i^j} = 0,\ l,m = 1,2,\dots,n$

(3.11) $\dfrac{\partial L_i^j}{\partial (\lambda)_i^j} = 0$

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concave, and all the constraints of $\leq$, including that of V_i^j if it were not an equality, convex, by Khun-Tucker sufficient second-order conditions.) Inequality constraints are linear and specified accordingly. SOC should hold for FOC to be valid for a maximum.

[6] We will concentrate on solutions with spontaneously positive values for these controls.

Looking at (3.5) and (3.6) separately, we observe that they form a system of nxn linear equations in the nxn $(\mu_l{}^m)_i{}^j$'s, l,m = 1,2,....,n, and that $(\mu_l{}^m)_i{}^j = 0$ for l≠i would solve it. We can write the two equations as:

$$0 = 1 - (\mu_i{}^j)_i{}^j + \sum_{m=1}^{n} (\mu_i{}^m)_i{}^j (1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m) \frac{\delta_i}{n-1} - (\mu_i{}^j)_i{}^j (1 - r_j$$

$$\prod_{\substack{k=1 \\ k \neq j}}^{n} s_k^j) \frac{\delta_i}{n-1}, \text{ or}$$

$$(3.12) (\mu_i{}^j)_i{}^j [1 + (1 - r_j \prod_{\substack{k=1 \\ k \neq j}}^{n} s_k^j) \frac{\delta_i}{n-1}] = 1 + \sum_{m=1}^{n} (\mu_i{}^m)_i{}^j (1 - r_m$$

$$\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m) \frac{\delta_i}{n-1}, \text{ and}$$

$$0 = -(\mu_r{}^s)_i{}^j + \sum_{m=1}^{n} (\mu_r{}^m)_i{}^j (1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m) \frac{\delta_r}{n-1} - (\mu_r{}^s)_i{}^j (1 - r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} s_k^s)$$

$$\frac{\delta_r}{n-1}, \text{ r≠i or s≠j, r,s = 1,2,...,n, or}$$

$$(3.13) (\mu_r{}^s)_i{}^j [1 + (1 - r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} s_k^s) \frac{\delta_r}{n-1}] = \sum_{m=1}^{n} (\mu_r{}^m)_i{}^j (1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m)$$

$$\frac{\delta_r}{n-1}, \text{ r≠i or s≠j, r,s = 1,2,...,n}$$

As the second equation holds for r=i if s≠j:

$$(3.14) (\mu_i{}^s)_i{}^j [1 + (1 - r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} s_k^s) \frac{\delta_i}{n-1}] = \sum_{m=1}^{n} (\mu_i{}^m)_i{}^j (1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m)$$

$$\frac{\delta_i}{n-1} \text{ s≠j, s = 1,2,...,n}$$

Then, from (3.12):

$$(3.15)\ (\mu_i^{\cdot j})_i^{\cdot j}\ [1\ +\ (1\ -\ r_j\prod_{\substack{k=1\\k\neq j}}^{n} s_k^j)\ \frac{\delta_i}{n-1}]\ -\ 1\ =\ (\mu_i^{\cdot s})_i^{\cdot j}\ [1\ +\ (1\ -\ r_s\prod_{\substack{k=1\\k\neq s}}^{n} s_k^s)$$

$$\frac{\delta_i}{n-1}]\ s\neq j,\ s\ =\ 1,2,...,n$$

The right hand-side is constant for all s. Replacing then $(\mu_i^{\cdot s})_i^{\cdot j}$, $s\neq j$, - after (3.15) - in (3.5):

$$(3.16)\quad 1\ =\ (\mu_i^{\cdot j})_i^{\cdot j}\ -\ \sum_{\substack{m=1\\m\neq j}}^{n}\ (\mu_i^{\cdot m})_i^{\cdot j}\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}\ =\ (\mu_i^{\cdot j})_i^{\cdot j}\ -\ -$$

$$\{(\mu_i^{\cdot j})_i^{\cdot j}\ [1\ +\ (1\ -\ r_j\prod_{\substack{k=1\\k\neq j}}^{n} s_k^j)\ \frac{\delta_i}{n-1}]\ -\ 1\}\ \sum_{\substack{m=1\\m\neq j}}^{n}\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}\ /\ [1$$

$$+\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}],\ or$$

$$(3.17)\quad (\mu_i^{\cdot j})_i^{\cdot j}\ =\ \{1\ -\ \sum_{\substack{m=1\\m\neq j}}^{n}\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}\ /\ [1\ +\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m$$

$$)\ \frac{\delta_i}{n-1}]\}\ /\ \{1\ -\ [1\ +\ (1\ -\ r_j\prod_{\substack{k=1\\k\neq j}}^{n} s_k^j)\ \frac{\delta_i}{n-1}]\sum_{\substack{m=1\\m\neq j}}^{n}\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}\ /$$

$$[1\ +\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)\ \frac{\delta_i}{n-1}]\}$$

$(\mu_i^{\cdot s})_i^{\cdot j}$ for $s\neq j$ can then be inferred from (3.15).

For any $r\neq i$, from (3.13) we deduct that:

$$(3.18)\,(\mu_r^{\cdot s})_i^{\cdot j}\ [1\ +\ (1\ -\ r_s\prod_{\substack{k=1\\k\neq s}}^{n} s_k^s)\ \frac{\delta_r}{n-1}]\ =\ (\mu_r^{\cdot m})_i^{\cdot j}\ [1\ +\ (1\ -\ r_m\prod_{\substack{k=1\\k\neq m}}^{n} s_k^m)$$

$$\frac{\delta_r}{n-1}]$$

Replacing in (3.6):

$$(3.19) \quad (\mu_r{}^s)_i^j = \sum_{\substack{m=1 \\ m \neq s}}^n (\mu_r{}^m)_i^j \left(1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^n s_k^m \right) \frac{\delta_r}{n-1} = (\mu_r{}^s)_i^j [1 + (1$$

$$- r_s \prod_{\substack{k=1 \\ k \neq s}}^n s_k^s) \frac{\delta_r}{n-1}] \sum_{\substack{m=1 \\ m \neq s}}^n (\mu_r{}^m)_i^j \left(1 - r_m \prod_{\substack{k=1 \\ k \neq m}}^n s_k^m\right) \frac{\delta_r}{n-1} \ / \ [1 + (1 - r_m$$

$$\prod_{\substack{k=1 \\ k \neq m}}^n s_k^m) \frac{\delta_r}{n-1}], \ r \neq i, \ r,s = 1,2,...,n$$

Then, it solves for:

$$(3.20) \, (\mu_r{}^s)_i^j = 0, \text{ for } r \neq i, \ r,s = 1,2,...,n$$

This implies that only the constraints relative to V_i^s, $s = 1,2,...,n$, are relevant for an optimisation problem of individual i – even though the others remain in the – equilibrium - background.

For interior solutions of s_i^j and r_i - $0 < s_i^j$, $r_i < 1$ -, the system would be completed with $(\eta^m)_i^j = 0$, $m \neq i$, $m = 1,2,...,n$, and $(\eta)_i^j = 0$. Due to (3.20), (3.7) – for interior solutions of x_r^i for $r \neq i$ - implies $(\lambda)_i^j = 0$; but then $r_i \prod_{\substack{k=1 \\ k \neq i}}^n s_k^i \ \frac{u_i{}'(x_i^i)}{1 - \delta_i} = 0$. We conclude that – if $u_i'(x_i^i)$ cannot be zero in the relevant range for x_i^i – that $r_i \prod_{\substack{k=1 \\ k \neq i}}^n s_k^i = 0$. For an interior solution for x_i^i (and some other x_i^j), r_i or s_k^i for at least one of the other players must be zero – the conditions would be incompatible with interior solutions, or even $s_k^i = 1$, for other players.

More interesting are the cases for which $s_i^j = 1$, $j \neq i$, $j = 1,2,...,n$, and $r_i = 1$, corresponding to pure strategies on such variables. Then:

$$(3.21) \quad \frac{\partial L_i^j}{\partial V_i^j} = 1 - (\mu_i^j)_i^j = 0 \text{ or } (\mu_i^j)_i^j = 1$$

(3.22) $\quad \dfrac{\partial L_i^j}{\partial V_r^s} = -(\mu_r^{\ s})_i^{\cdot j} = 0$, or $(\mu_r^{\ s})_i^{\cdot j} = 0$ r≠i or s≠j, r,s = 1,2,...,n

(3.23) $\quad \dfrac{\partial L_i^j}{\partial s_i^s} = \displaystyle\sum_{\substack{m=1 \\ m\neq s}}^{n} \sum_{l=1}^{n} (\mu_l^{\ m})_i^{\cdot j} \left[\dfrac{u_l(x_l^m)}{1-\delta_l} - (d_l^m + \dfrac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k\neq m}}^{n} V_l^k) \right] -$

$(\eta^s)_i^{\cdot j} = 0$, s≠i, s = 1,2,...,n

(3.24) $\dfrac{\partial L_i^j}{\partial r_i} = \displaystyle\sum_{l=1}^{n} (\mu_l^{\ i})_i^{\cdot j} \left[\dfrac{u_l(x_l^i)}{1-\delta_l} - (d_l + \dfrac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k\neq i}}^{n} V_l^k) \right] - (\eta)_i^{\cdot j} = 0$

(3.25) $V_l^{\ m} = \dfrac{u_m(x_l^m)}{1-\delta_m}$, l,m = 1,2,...,n

(3.7) would hold, but with no added insights. We now have that the multipliers take a sort of canonical form, with $(\mu_i^{\ j})_i^{\cdot j}$ taking the value 1, and all others 0.

(3.21) and (3.22), from $\dfrac{\partial L_i^j}{\partial s_i^s} = 0$, imply:

(3.26) $\quad (\eta^j)_i^{\cdot j} = 0$, and

(3.27) $\dfrac{u_i(x_i^j)}{1-\delta_i} - (d_i^j + \dfrac{\delta_i}{n-1} \displaystyle\sum_{\substack{k=1 \\ k\neq j}}^{n} V_i^k) = (\eta^s)_i^{\cdot j}$ s≠i,j, s = 1,2,...,n

From $\dfrac{\partial L_i^j}{\partial r_i} = 0$:

(3.28) $(\eta)_i^{\cdot j} = 0$ if i≠j and

(3.29) $\dfrac{u_i(x_i^i)}{1-\delta_i} - (d_i + \dfrac{\delta_i}{n-1} \displaystyle\sum_{\substack{k=1 \\ k\neq i}}^{n} V_i^k) = (\eta)_i^{\ i}$

The system applying to round j and player i – for given values of opponents' strategies and his own at other rounds than j - is indeterminate. We claimed, that only contemporaneous (effective, i.e.,

other than those relative to the V_l^m's) control conditions are restrictive – condition (3.27) should therefore be superfluous; in fact, its addition just adds new unknowns - $(\eta^s)_i^j$ - and does not help (at this point) in determination.

Mixed strategies with respect to r_i, but pure ones with respect to s_i^j (with acceptance) $s_i^j = 1$, $j \neq i$, $j = 1,2,...,n$, would add to (3.5) to (3.11), $(\eta)_i^j = 0$:

$$(3.30) \quad \frac{\partial L_i^j}{\partial V_i^j} = 1 - (\mu_i^j)_i^j + \sum_{\substack{m=1 \\ m \neq j}}^{n} (\mu_i^m)_i^j (1 - r_m) \frac{\delta_i}{n-1} = 0 = 1 -$$

$$(\mu_i^j)_i^j + \sum_{m=1}^{n} (\mu_i^m)_i^j (1 - r_m) \frac{\delta_i}{n-1} - (\mu_i^j)_i^j (1 - r_i) \frac{\delta_i}{n-1}$$

$$(3.31) \quad \frac{\partial L_i^j}{\partial V_r^s} = - (\mu_r^s)_i^j + \sum_{\substack{m=1 \\ m \neq s}}^{n} (\mu_r^m)_i^j (1 - r_m) \frac{\delta_r}{n-1} = 0 =- (\mu_r^s)_i^j$$

$$+ \sum_{m=1}^{n} (\mu_r^m)_i^j (1 - r_m) \frac{\delta_r}{n-1} - (\mu_r^s)_i^j (1 - r_r) \frac{\delta_r}{n-1}, \; r \neq i \text{ or } s \neq j, \; r,s = $$
1,2,...,n

$$(3.32) \quad \frac{\partial L_i^j}{\partial s_i^s} = \sum_{\substack{m=1 \\ m \neq s}}^{n} \sum_{l=1}^{n} (\mu_l^m)_i^j r_m \left[\frac{u_l(x_l^m)}{1 - \delta_l} - (d_l^m + \frac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k \neq m}}^{n} V_l^k) \right] - $$

$(\eta^s)_i^j = 0$, $s \neq i$, $s = 1,2,...,n$

$$(3.33) \quad \frac{\partial L_i^j}{\partial r_i} = \sum_{l=1}^{n} (\mu_l^i)_i^j \left[\frac{u_l(x_l^i)}{1 - \delta_l} - (d_l + \frac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k \neq i}}^{n} V_l^k) \right] = 0$$

$$(3.34) \; V_l^m = r_m \frac{u_m(x_l^m)}{1 - \delta_m} + (1 - r_m) (d_l + \frac{\delta_l}{n-1} \sum_{\substack{k=1 \\ k \neq m}}^{n} V_l^k), \; l,m = $$

1,2,...,n

(3.30) and (3.31) imply, at given r_m's and δ_r's, a system of n equations and n unknowns, the $(\mu_l^m)_i^j$, l,m = 1,2,...,n. It solves – using (3.17), (3.15) and (3.20) - for:

$$(3.35) \quad (\mu_i^j)_i^j = \{1 - \sum_{\substack{m=1 \\ m \neq j}}^{n} (1 - r_m) \frac{\delta_i}{n-1} \,/\, [1 + (1 - r_m) \frac{\delta_i}{n-1}]\} \,/\, \{1$$

$$- [1 + (1 - r_j) \frac{\delta_i}{n-1}] \sum_{\substack{m=1 \\ m \neq j}}^{n} (1 - r_m) \frac{\delta_i}{n-1} \,/\, [1 + (1 - r_m) \frac{\delta_i}{n-1}]\}$$

$$(3.36) \ (\mu_i^m)_i^j = ([1 + (1 - r_j) \frac{\delta_i}{n-1}] \{1 - \sum_{\substack{m=1 \\ m \neq j}}^{n} (1 - r_m) \frac{\delta_i}{n-1} \,/\, [1 + (1$$

$$- r_m) \frac{\delta_i}{n-1}]\} \,/\, \{1 - [1 + (1 - r_j) \frac{\delta_i}{n-1}] \sum_{\substack{m=1 \\ m \neq j}}^{n} (1 - r_m) \frac{\delta_i}{n-1} \,/\, [1 + (1 -$$

$$r_m) \frac{\delta_i}{n-1}]\} - 1) \,/\, [1 + (1 - r_m) \frac{\delta_i}{n-1}], \ m \neq j, \ m = 1,2,...,n,$$

$$(3.37) (\mu_l^m)_i^j = 0, \ l \neq i, \ l,m = 1,2,...,n.$$

Then, (3.32), from $\dfrac{\partial L_i^j}{\partial s_i^s} = 0$, implies:

$$(3.38) \sum_{\substack{m=1 \\ m \neq s}}^{n} (\mu_i^m)_i^j \, r_m \, [\frac{u_i(x_i^m)}{1 - \delta_i} - (d_i^m + \frac{\delta_i}{n-1} \sum_{\substack{k=1 \\ k \neq m}}^{n} V_i^k)] = (\eta^s)_i^j, \ s \neq i, \ s$$

$$= 1,2,...,n$$

where (3.35) and (3.36) could be replaced.

(3.33) reverts to:

$$(3.39) \frac{u_i(x_i^i)}{1 - \delta_i} = d_i + \frac{\delta_i}{n-1} \sum_{\substack{k=1 \\ k \neq i}}^{n} V_i^k$$

and (3.34) holds.

(3.39), replacing in the definition of V_i^i also requires:

$$(3.40) \quad V_i^{\ i} = \frac{u_i(x_i^i)}{1 - \delta_i}$$

3.3 Nash Equilibrium

A Nash equilibrium will involve the coincidence of value solutions for $V_i^{\ j}$, $x_i^{\ j}$, $s_i^{\ j}$, and r_i from the various, nxn, problems.

Multipliers are specific to each particular programme – and therefore they appeared indexed by $(.)_i^{\ j}$.

With pure strategies, (3.27) occurs at n-1 problems of each individual i. Then, it implies, with (3.25):

$$(3.41) \frac{u_i(x_i^j)}{1 - \delta_i} \ - \ (d_i^{\ j} \ + \ \frac{\delta_i}{n - 1} \ \sum_{\substack{k=1 \\ k \neq j}}^{n} \ \frac{u_i(x_i^k)}{1 - \delta_i}) \ = \ (\eta^s)_i^{\ j}, \ \text{ if } s \neq i,j, \ s,i,j$$

$=1,2,....,n$

Likewise, (3.29) becomes:

$$(3.42) \frac{u_i(x_i^i)}{1 - \delta_i} \ - \ (d_i \ + \ \frac{\delta_i}{n - 1} \ \sum_{\substack{k=1 \\ k \neq i}}^{n} \ \frac{u_i(x_i^k)}{1 - \delta_i}) \ = \ (\eta)_i^{\ i}$$

Due to the structure of the equations, the system remains indeterminate. So one could expect that players affect each other's multipliers to their own benefit and to the extent of their own ability within the game's protocol. Also, even if condition (3.41) would normally be redundant, it should be observed – with non-negative multipliers in an optimum. One can assume that player i is going to force $(\eta)_i^{\ i}$ up – maximising his share, $x_i^{\ i}$, according to condition (3.42) -, the multiplier associated with r_i; and/or $(\eta^s)_l^{\ i}$'s, $l,s \neq i$, $l = 1,2,...,n$, to a minimum within its allowable range – i.e., i will press $(\eta^s)_l^{\ i}$ – the multipliers associated to the $s_l^{\ m}$'s at round i, minimising $x_l^{\ i}$ (for given policies of other players) - in all other player's, l, problem i to zero. That is to say, decreases all others' shares in x^i – relative to his – but guaranteeing condition (3.41) to be observed for all other players

within the allowable range, i.e., non-negative $(\eta^s)_l^{i}$'s[7] - that measure the incremental welfare effect for player l at round i of his acquiescence ability at later settlements. Notice that forcing $(\eta^s)_l^{i}$ to zero in other players' problems implies $(\eta)_i^{i}$ being maximised and those conditions *per se* provide a corner that assures an interior equilibrium solution. Then, (1.1) and:

$$(3.43)\ \frac{u_i(x_i^j)}{1 - \delta_i} = d_i^{j} + \frac{\delta_i}{n-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{u_i(x_i^k)}{1 - \delta_i},\ j{\neq}i,\ i,j =1,2,....,n$$

provide a full solution to the game.

With mixed strategies, the indeterminacy prevails. With the reasoning applied to $(\eta^s)_i^{j}$ replicated to the current problem, we conclude that an (semi-)internal solution follows:

$$(3.44)\ \frac{u_i(x_i^j)}{1 - \delta_i} = d_i^{j} + \frac{\delta_i}{n-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} V_i^k\ \ j{\neq}i,\ i,j =1,2,....,n$$

$$(3.45)\ V_i^{i} = \frac{u_i(x_i^i)}{1 - \delta_i} = d_i + \frac{\delta_i}{n-1} \sum_{\substack{k=1 \\ k \neq i}}^{n} V_i^k\ ,\ i =1,2,....,n$$

With (3.34) and (1.1) – or the equivalent restriction from all the problems -, equilibrium values for x_i^{j}, V_i^{j}, and r_i are obtainable.

A solution can then be obtained with also the other definitions – exhibiting the properties stated in Martins (2006).

As long as $(\eta)_i^{i}$ is (can be) positive (d_i is very small compared to the n alternative equilibrium shares for player i) in pure strategies, a better solution should be attainable - pure strategies should always outperform mixed ones.

4 SIMULTANEOUS GAMES

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[7] Required by Khun-Tucker conditions for a maximum.

4.1 The Player's Problem

Assume an n-persons synchronised offers game.

r_i is the probability with which player i makes an offer at each round of negotiations, $(1 - r_i)$ the one with which he decides not to. One and only one offer is going to be heard; j's offer will be the one considered iff he makes an offer but not the other players, which occurs with probability:

$$(4.1) \quad r_j \prod_{\substack{k=1 \\ k \neq j}}^{n} (1 - r_k)$$

A – and only one - vector x^j is, thus either going to be accepted unanimously, generating pay-off $\dfrac{u_i(x_i^j)}{1 - \delta_i}$ for each player i; or rejected with the game reinitiating next period with a delay involving losses d_i^j (that eventually differ according to the effectively offering, rejected, party) for player i – which therefore gets then pay-off $d_i^j + \delta_i V_i$, where V_i is the value of the game for player i.

If no offer arises, which occurs with probability $\prod_{k=1}^{n}(1 - r_k)$ at each round, i gets payoff $d_i + \delta_i V_i$ – he obtains d_i in the period and bargaining re-initiates next period.

If more than one player make offers, the game re-starts with no delay – each player maintaining his expectations, V_i.

Each player i will solve the problem:

$$(4.2) \quad \underset{\substack{V_1,\dots,V_2,\dots,V_n; \\ r_i; \; x_1^i,\dots,x_i^i,\dots,x_n^i; \\ s_i^1,\dots,s_i^{i-1},s_i^{i+1},\dots,s_i^n}}{\text{Max}} V_i \; \text{s.t.:}$$

$$(4.3) \; V_l = \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \; [\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \; \frac{u_l(x_l^m)}{1 - \delta_l} \; + (1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m) \; (d_l^m +$$

$$\delta_l \, V_l)] + \prod_{k=1}^{n} (1 - r_k) \; (d_l + \delta_l \, V_l) + [1 - \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) - \prod_{k=1}^{n} (1 - r_k)]$$

$$V_l, \; l = 1,2,...,n$$

$$\sum_{l=1}^{n} x_l^i = 1$$

$$x_l^i \geq 0, \; l = 1,2,...,n;$$

$$0 \leq r_i \leq 1; \; 0 \leq s_i^m \leq 1, \; m \neq i, \; m = 1,2,...,n$$

Given r_m, $m \neq i$, $m = 1,2,\ldots,n$;

x_l^m, $m \neq i$, $l,m = 1,2,\ldots,n$;

s_l^m, $l \neq i,m$, $l,m = 1,2,\ldots,n$.

Equation (4.3) can be simplified to:

$$(4.4) \; V_l = \{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \; [\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \; \frac{u_l(x_l^m)}{1 - \delta_l} \; + (1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m) \; d_l^m]$$

$$+ \prod_{k=1}^{n} (1 - r_k) \; d_l \} \; / \; \{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \; [1 - \delta_l \, (1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m)] + (1 - \delta_l)$$

$$\prod_{k=1}^{n} (1 - r_k) \}, \; l = 1,2,...,n$$

4.2 First Order Conditions

Player i is going to face the Lagrangean:

(4.5) $\underset{\substack{V_1,...,V_i,...,V_n; \\ r_i; x_1^i,...,x_i^i,...,x_n^i; \\ s_i^1,...,s_i^{i-1},s_i^{i+1},...,s_i^n; \\ (\mu_l)_i,...,(\mu_i)_i,...,(\mu_n)_i; \\ (\lambda)_i; \\ (\eta^1)_i,...,(\eta^{i-1})_i,(\eta^{i+1})_i,...,(\eta^n)_i; (\eta)_i}}{\text{Max}}$

$$L_i = V_i + \sum_{l=1}^{n} (\mu_l)_i \left(- V_l \left\{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \left[1 - \delta_l \left(1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) \right] + \right. \right.$$

$$(1 - \delta_l) \prod_{k=1}^{n} (1 - r_k) \right\} + \left\{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \left[\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \ \frac{u_l(x_l^m)}{1 - \delta_l} + \left(1 - \right. \right. \right.$$

$$\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) d_l^m \right] + \prod_{k=1}^{n} (1 - r_k) \ d_l \right\} \right) + (\lambda)_i \left(1 - \sum_{l=1}^{n} x_l^i \right) + (\eta)_i (1 - r_i) +$$

$$\sum_{\substack{m=1 \\ m \neq i}}^{n} (\eta^m)_i (1 - s_i^m)$$

$x_l^i \geq 0, \ l = 1,2,...,n;$

$(\lambda)_i \geq 0; \ (\mu_l)_i \geq 0, \ l = 1,2,...,n$

$(\eta)_i (1 - r_i) = 0, \ r_i \geq 0, \ (\eta)_i \geq 0;$

$(\eta^m)_i (1 - s_i^m) = 0, \ s_i^m \geq 0, \ (\eta^m)_i \geq 0, \ m \neq i, \ m = 1,2,...,n$

Then for optimal solutions with positive values of x_i^j, s_i^j and r_i we have:

$$(4.6) \quad \frac{\partial L_i}{\partial V_i} = 1 - (\mu_i)_i \left\{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \left[1 - \delta_i \left(1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) \right] + \left(1 - \right. \right.$$

$$\delta_i) \prod_{k=1}^{n} (1 - r_k) \right\} = 0$$

$$(4.7) \quad \frac{\partial L_i}{\partial V_r} = -(\mu_r)_i \left\{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) \left[1 - \delta_r \left(1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) \right] + (1 - \delta_r) \right.$$

$$\left. \prod_{k=1}^{n} (1 - r_k) \right\} = 0, \ r \neq i, \ r = 1,2,\dots,n$$

$$(4.8) \quad \frac{\partial L_i}{\partial x_r^i} = (\mu_r)_i \, r_i \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - r_k) \prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i \ \frac{u_r{}'(x_r^i)}{1 - \delta_r} - (\lambda)_i = 0, \ r = 1,2,\dots,n$$

$$(4.9) \quad \frac{\partial L_i}{\partial s_i^s} = \sum_{l=1}^{n} (\mu_l)_i \left[-V_l \, \delta_l \, r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} (1 - r_k) \prod_{\substack{k=1 \\ k \neq s}}^{n} s_k^s + r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} (1 - r_k) \right.$$

$$\left. \prod_{\substack{k=1 \\ k \neq s}}^{n} s_k^s \left(\frac{u_l(x_l^s)}{1 - \delta_l} - d_l^s \right) \right] / s_i^s - (\eta^s)_i = 0, \ s \neq i, \ s = 1,2,\dots,n$$

$$(4.10) \quad \frac{\partial L_i}{\partial r_i} = \sum_{l=1}^{n} (\mu_l)_i \left(-V_l \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - r_k) \left[1 - \delta_l \left(1 - \prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i \right) \right] - \sum_{\substack{m=1 \\ m \neq i}}^{n} r_m \right.\right.$$

$$\prod_{\substack{k=1 \\ k \neq m \\ k \neq i}}^{n} (1 - r_k) \left[1 - \delta_l \left(1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) \right] - (1 - \delta_l) \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - r_k) \Bigg\} + \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - r_k) \left[\rule{0pt}{2.5ex}\right.\right.$$

$$\prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i \ \frac{u_l(x_l^i)}{1 - \delta_l} + \left(1 - \prod_{\substack{k=1 \\ k \neq i}}^{n} s_k^i \right) d_l^i \right] - \sum_{\substack{m=1 \\ m \neq i}}^{n} r_m \prod_{\substack{k=1 \\ k \neq m \\ k \neq i}}^{n} (1 - r_k) \left[\prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \ \frac{u_l(x_l^m)}{1 - \delta_l} \right.$$

$$+ \left(1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m \right) d_l^m \right] - \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - r_k) \, d_l \Bigg\}) - (\eta)_i = 0$$

$$(4.11) \quad \frac{\partial L_i}{\partial (\mu_l)_i} = 0, \ l = 1,2,\dots,n$$

$$(4.12) \quad \frac{\partial L_i}{\partial (\lambda)_i} = 0, \ m = 1,2,\dots,n$$

Due to (4.7), optimal solutions will imply for each i, either of the two factors of:

(4.13) $(\mu_r)_i$ $\{ \sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k)$ $[1 - \delta_r (1 - \prod_{\substack{k=1 \\ k \neq m}}^{n} s_k^m)] + (1 - \delta_r)$

$\prod_{k=1}^{n} (1 - r_k) \} = 0$, r≠i, r = 1,2,…,n

will be zero.

For interior solutions of s_i^j and r_i, the system would be completed with $(\eta^m)_i = 0$, m≠i, m = 1,2,...,n, and $(\eta)_i = 0$.

Mixed strategies with respect to r_i, but pure ones with respect to s_i^j (with acceptance) $s_i^j = 1$, j≠i, j = 1,2,...,n, would add to (4.6) to (4.12), $(\eta)_i = 0$:

(4.14) $\quad 1 = (\mu_i)_i [\sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) + (1 - \delta_i) \prod_{k=1}^{n} (1 - r_k)]$

(4.15) $\quad (\mu_r)_i [\sum_{m=1}^{n} r_m \prod_{\substack{k=1 \\ k \neq m}}^{n} (1 - r_k) + (1 - \delta_r) \prod_{k=1}^{n} (1 - r_k)] = 0$, $(\mu_r)_i =$

0, r≠i, r = 1,2,…,n

because the second factor must be positive. Then:

(4.16) $\quad \dfrac{\partial L_i}{\partial s_i^s} = \sum_{l=1}^{n} (\mu_l)_i [- V_l \delta_l r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} (1 - r_k) + r_s \prod_{\substack{k=1 \\ k \neq s}}^{n} (1 - r_k) ($

$\dfrac{u_l(x_l^s)}{1 - \delta_l} - d_l^s)] - (\eta^s)_i = 0$, s≠i, s = 1,2,…,n

(4.17) $\dfrac{\partial L_i}{\partial r_i}$ $=$ $-$ V_i $[\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)$ $-$ $\displaystyle\sum_{\substack{m=1\\m\neq i}}^{n} r_m \prod_{\substack{k=1\\k\neq m\\k\neq i}}^{n}(1-r_k)$ $-$ $(1-\delta_i)$

$\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)]$ $+$ $\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)\dfrac{u_i(x_i^i)}{1-\delta_i}$ $-$ $\displaystyle\sum_{\substack{m=1\\m\neq i}}^{n} r_m \prod_{\substack{k=1\\k\neq m\\k\neq i}}^{n}(1-r_k)$ $\dfrac{u_i(x_i^m)}{1-\delta_i}$ $-$

$\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)\, d_i = 0,$ and

(4.18) $V_l = [\displaystyle\sum_{m=1}^{n} r_m \prod_{\substack{k=1\\k\neq m}}^{n}(1-r_k)$ $\dfrac{u_l(x_l^m)}{1-\delta_l}$ $+$ $\displaystyle\prod_{k=1}^{n}(1-r_k)\,d_l]$ $/$ $[\displaystyle\sum_{m=1}^{n} r_m$

$\displaystyle\prod_{\substack{k=1\\k\neq m}}^{n}(1-r_k)$ $+$ $(1-\delta_l)\displaystyle\prod_{k=1}^{n}(1-r_k)],\, l = 1,2,\ldots,n$

From (4.16) and (4.14):

(4.19) $r_s \displaystyle\prod_{\substack{k=1\\k\neq s}}^{n}(1-r_k)$ $[$ $-$ $V_i\,\delta_i$ $+$ $\dfrac{u_i(x_i^s)}{1-\delta_i}$ $-$ d_i^s $]$ $=$ $(\eta^s)_i$ $[\displaystyle\sum_{m=1}^{n} r_m$

$\displaystyle\prod_{\substack{k=1\\k\neq m}}^{n}(1-r_k)$ $+$ $(1-\delta_i)\displaystyle\prod_{k=1}^{n}(1-r_k)],$ $s\neq i,\, s = 1,2,\ldots,n$

Departing from (4.17):

(4.20) V_i $[\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)$ $-$ $\displaystyle\sum_{\substack{m=1\\m\neq i}}^{n} r_m \prod_{\substack{k=1\\k\neq m\\k\neq i}}^{n}(1-r_k)$ $-$ $(1-\delta_i)$ $\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)]$ $=$

$\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)\dfrac{u_i(x_i^i)}{1-\delta_i}$ $-$ $\displaystyle\sum_{\substack{m=1\\m\neq i}}^{n} r_m \prod_{\substack{k=1\\k\neq m\\k\neq i}}^{n}(1-r_k)$ $\dfrac{u_i(x_i^m)}{1-\delta_i}$ $-$ $\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k)\,d_i$

Divided by $\displaystyle\prod_{\substack{k=1\\k\neq i}}^{n}(1-r_k):$

$$(4.21) \quad V_i \left(\delta_i - \sum_{\substack{m=1 \\ m \neq i}}^{n} \frac{r_m}{1 - r_m} \right) = \frac{u_i(x_i^i)}{1 - \delta_i} - \sum_{\substack{m=1 \\ m \neq i}}^{n} \frac{r_m}{1 - r_m} \frac{u_i(x_i^m)}{1 - \delta_i} - d_i$$

with (4.18) for i divided by $\prod_{k=1}^{n} (1 - r_k)$:

$$(4.22) \quad V_i \left(\sum_{m=1}^{n} \frac{r_m}{1 - r_m} + 1 - \delta_i \right) = \sum_{m=1}^{n} \frac{r_m}{1 - r_m} \frac{u_i(x_i^m)}{1 - \delta_i} + d_i, \quad i = 1,2,\ldots,n$$

Subtracting both sides of (4.21) from those of (4.22):

$$(4.23) \quad V_i \left(\frac{r_i}{1 - r_i} + 1 \right) = \left(\frac{r_i}{1 - r_i} + 1 \right) \frac{u_i(x_i^i)}{1 - \delta_i}$$

and therefore

$$(4.24) \quad V_i = \frac{u_i(x_i^i)}{1 - \delta_i}$$

4.3 Equilibrium

A Nash equilibrium will involve the coincidence of value solutions for V_i, x_i^j, s_i^j, and r_i from the various, n, problems.

Multipliers are specific to each particular programme – and therefore they appeared indexed by $(.)_i$. As before, we sort to the conclusion that $(\eta^s)_i$ will be pressed to 0. Then (4.19) implies:

$$(4.25) \quad \frac{u_i(x_i^s)}{1 - \delta_i} = d_i^s + \delta_i V_i, \quad s \neq i, \ i,s = 1,2,\ldots,n$$

As (4.24) requires:

$$(4.26) \quad V_i = \frac{u_i(x_i^i)}{1 - \delta_i}, \quad i = 1,2,\ldots,n$$

With (1.1) – or the equivalent from all the problems - a solution for the x_i^j's – and therefore the V_i's - can be obtained independently from:

(4.27) $\dfrac{u_i(x_i^m)}{1 - \delta_i} = d_i{}^m + \delta_i \dfrac{u_i(x_i^i)}{1 - \delta_i}$, m≠i, i,m = 1,2,…,n

A solution for the r_j's can then be retrieved from (4.21) with the new equalities:

(4.28) $\dfrac{u_i(x_i^i)}{1 - \delta_i} (\delta_i - \displaystyle\sum_{\substack{m=1 \\ m \neq i}}^{n} \dfrac{r_m}{1 - r_m}) = \dfrac{u_i(x_i^i)}{1 - \delta_i} - \displaystyle\sum_{\substack{m=1 \\ m \neq i}}^{n} \dfrac{r_m}{1 - r_m} [d_i{}^m + \dfrac{u_i(x_i^i)}{1 - \delta_i}$

$\delta_i]$ - d_i, i = 1,2,…,n, or

(4.29) $u_i(x_i^i)$ - $d_i = \displaystyle\sum_{\substack{m=1 \\ m \neq i}}^{n} \dfrac{r_m}{1 - r_m} [d_i{}^m - u_i(x_i^i)]$, i = 1,2,…,n

With x_i^i's, r_m's can be inferred from the last equation system – see Martins (2006).

5 CONCLUSIONS

It was shown how different protocols of the bargaining over the distribution of an exogenously fixed asset can be mathematically programmed. On the one hand, the individuals' problems, were adequately stated, as well as insufficiency of FOC to originate a solution. In alternate offers, the indeterminacy remained even with the introduction of intertemporal optimisation restrictions. On the other, additional conditions qualifying the known solutions were introduced and rationalised.

One concluded that equilibrium determination in splitting games – at least in those inspected - can be adequately interpreted as the result of each individual's choice, when he is deciding the own proposition, of the corner - guaranteeing non-negative multipliers of the acceptance probabilities of other players also towards other players' offers - that is most advantageous to him over the optimal set provided by the standard FOC's. In those games, such rule generated the possibility of a unique interior solution.

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